

## ASYMPTOTICS FOR MOMENTS OF HIGHER RANKS

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ABSTRACT. Bringmann, Mahlburg, and Rhoades have found asymptotic expressions for all moments of the partition statistics rank and crank. In this work we extend their methods to higher ranks. The  $T$ -rank, introduced by Garvan, for odd integers  $T = 3$  is a natural generalization of the rank ( $T = 3$ ) and crank ( $T = 1$ ).

## 1. INTRODUCTION AND STATEMENT OF RESULTS

In the theory of integer partitions the partition statistics "rank" (defined by Dyson) and "crank" (defined by Andrews and Garvan) play a fundamental role.

The rank was first introduced by Dyson [Dys44] in the attempt to explain the Ramanujan congruences for the partition function from a combinatorial point of view.

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

Indeed, Atkin and Swinnerton-Dyer [ASD54] proved that the rank explains the first two congruences. Dyson already observed that the third congruence could not be explained by the rank, which led him to speculate about the existence of a different partition statistic, which he termed the "crank", which would explain all three congruences. This statistic was later found by Andrews and Garvan [AG88].

While the combinatorial definitions of the rank and the crank are very different and do not allow for an immediate generalization, Garvan [Gar94] found a generalization by looking at generating functions. For an odd positive integer  $T$ , he defined the numbers  $N_T(m, n)$  by the following series

$$\sum_{n=0}^{\infty} N_T(m, n) q^n = \frac{1}{(q)_{\infty}} \sum_{n=1}^{\infty} (-1)^{n-1} q^{\frac{n}{2}(Tn-1)+|m|n} (1 - q^n),$$

where the  $q$ -Pochhammer symbol is defined by  $(x; q)_n := \prod_{k=1}^n (1 - xq^k)$ . Then,  $N_1(m, n)$  is the number of partitions of  $n$  having crank  $m$  and  $N_3(m, n)$  is the number of partitions of  $n$  having rank  $m$ . Although  $N_T$  is not the counting function for a partition statistic in a strict sense, Garvan [Gar94] also found combinatorial interpretations of the numbers  $N_T(m, n)$ .

The rank and crank received renewed interest only recently when Atkin and Garvan [AG03] defined rank and crank moments and Andrews [And08] discovered a combinatorial meaning of these moments.

We now define moments not only for the rank and crank but also for the  $T$ -ranks introduced by Garvan. For a positive odd integer  $T$  and a positive integer  $r$  we define

$$m_T^r(n) := \sum_{m=-n}^n m^r N_T(m, n).$$

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Andrews [And08] related his smallest parts function to rank and crank moments. More precisely, if we let  $\text{spt}(n)$  denote the total number of smallest parts counted in all partitions of  $n$ , then

$$\text{spt}(n) = \frac{1}{2} (m_1^2(n) - m_3^2(n)).$$

In [Gar10] Garvan raised the question whether  $m_1^r(n) > m_3^r(n)$  holds for all positive even integers  $r$ . In [BM09], Bringmann and Mahlburg realized that one could prove Garvan's conjecture (for  $n$  large enough), if one knew precise asymptotics of  $m_1^r(n)$  and  $m_3^r(n)$ . Indeed, Bringmann and Mahlburg found such asymptotic expressions for  $m_1^r(n)$  and  $m_3^r(n)$  and were able to prove that Garvan's conjecture holds in the cases  $r = 2, 4$  for sufficiently large  $n$ . After that, Bringmann, Mahlburg, and Rhoades in [BMR11] succeeded in proving asymptotics for all even  $r$ , thus settling Garvan's conjecture in the limit case. In [Gar11], Garvan proved his conjecture for all  $n$  and  $r$  by finding a combinatorial interpretation of the difference of the rank and crank moments in terms of higher spt-functions.

In this work, we will address the same questions for the moments of higher rank functions  $N_T$ . This means that our first goal is to derive asymptotic formulas for the moments of the  $T$ -ranks. Our second objective is to prove an analogue of Garvan's conjecture.

This research was part of the authors PhD thesis [Wal12]. At first, this was not motivated mainly by combinatorics but rather aimed at developing further the methods in the context of automorphic forms, which are used to derive the asymptotic formulas. However, after this work was finished, also a combinatorial interpretation of moments of higher ranks was found by Dixit and Yee [DY12]. Their interpretation gives Garvan's conjecture in the general case. Furthermore, our main theorem can be used to find asymptotic formulas for their "generalized higher spt-functions".

We now briefly outline our approach behind finding the asymptotic formulas for  $m_T^r(n)$ . The general idea is to apply the Circle Method to the generating function

$$M_T^r(q) := \sum_{n=0}^{\infty} m_T^r(n) q^n = \sum_{n=0}^{\infty} \sum_{m=-n}^n m^r N_T(m, n) q^n.$$

The philosophy of the Circle Method is as follows. The generating function  $M_T^r(q)$  defines a holomorphic function on the unit circle  $|q| < 1$  with singularities at the boundary. If one is able to find nice enough expressions for the shape of these singularities in the neighborhood of all roots of unity, one can use Cauchy's integral formula to determine asymptotic expressions for  $m_T^r(n)$ .

The proof in [BMR11] relies on the Circle Method combined with complicated recurrence relations for the rank and the crank. As a consequence, this approach can not be applied to the case of higher ranks. However, in [BMR12], Bringmann, Mahlburg, and Rhoades, using a new approach, improved upon their previous work and found asymptotic formulas for all rank and crank moments with error terms which are as small as one could hope for by using the Circle Method. The idea behind their new approach is the insight that it is easier to not study the generating functions  $M_T^r(q)$  individually, but rather a two-variable generating function involving  $M_T^r(q)$  for all  $r$  at the same time. In fact, it turns out that there is a two-variable function  $\mathcal{M}_T(u, q)$  whose Taylor coefficients turn out to be  $M_T^r(q)$ . Moreover, this generating function is a Jacobi form in the crank case (i.e.  $T = 1$ ) and a mock Jacobi form in the rank case (i.e.  $T = 3$ ). Now the idea is simple: One determines explicit transformation formulas for the (mock) Jacobi forms, uses an asymptotic Taylor expansion to obtain asymptotic formulas for  $M_T^r(q)$  for all  $r$  in the neighborhood of roots of unity and then applies the Circle Method.

The generalization of this idea to the higher ranks case, however, is not straightforward but some complications occur. Indeed, for  $T = 1$  the function  $\mathcal{M}_1(u, q)$  is easily seen to be a Jacobi form. For

$T = 3$ , we only obtain a mock Jacobi form<sup>1</sup>. As a consequence the determination of the explicit transformation laws becomes a rather technical issue. For  $T > 3$  we have to deal with even more complicated expressions. Luckily, this turns out to be challenging only from a notational point of view. However, a more serious issue arises for  $T > 3$ . While for  $T = 1$  we obtain a Jacobi form, for  $T > 1$  we no longer have true but only mock Jacobi transformation laws. The obstruction to modularity in the case  $T = 3$ , however, turns out to not affect the asymptotic behavior of  $M_T^r(q)$  at roots of unity significantly. For  $T > 3$  this is no longer true and we have to take into account also the contribution from the obstruction to modularity.

To state our main theorems, we need to introduce some notation. For positive integers  $a, b, c$ , we define the constants  $\kappa(a, b, c)$  and  $\kappa^*(a, b, c)$  as in equation (9) and (10) on page 6. Furthermore, we let  $K_k(n)$  and  $K_{\sigma, \rho, l; k}(n)$  denote the Kloosterman sums in equation (38) on page 24 and in equation (39) on page 25. We write  $I$  for the modified Bessel function and define  $\mathcal{I}$  to be an integral over a modified Bessel function as in equation (37) on page 22. Finally, the notation  $(\ )_+$  means that we only include the expression if the value in the parenthesis is bigger than 0, and regard it as 0 else. Now our main theorem reads as follows:

**Theorem A.** *Let  $T < 24$  be an odd integer and  $r$  an even integer. Then, we have*

$$\begin{aligned} m_T^r(n) = & 2\pi \sum_{k \leq \sqrt{n}} \frac{K_k(n)}{k} \sum_{2a+2b+2c=r} \kappa(a, b, c) (kT)^a (24n-1)^{-\frac{3}{4}+\frac{a}{2}+c} I_{-\frac{3}{2}+a+2c} \left( \frac{\pi\sqrt{24n-1}}{6k} \right) \\ & + 2\pi \sum_{\substack{\sigma|T \\ t \neq 0}} \sum_{t=-\frac{T-1}{2}}^{\frac{T-1}{2}} \sum_{\substack{\rho=-\frac{T-1}{2} \\ \ell=0}}^{\frac{T-1}{2}} \sum_{\substack{0 < k \leq \sqrt{n} \\ (k, T)=\sigma}} \sum_{l=0}^{\frac{k}{\sigma}-1} \frac{K_{\sigma, \rho, l; k}(n)}{k} \sum_{2a+(2b+1)+c=r} \kappa^*(a, b, c) k^{b-\frac{1}{2}} T^{b-\frac{1}{2}} \sigma^{c+\frac{1}{2}} \left( 2n - \frac{1}{12} \right)^{\frac{a+c}{2}-\frac{1}{4}} \\ & \times \left( \frac{1}{12} - \frac{\sigma^2}{T^3} \left( \rho^2 + \frac{T^2}{4} - |\rho|T \right) \right)_+^{\frac{3}{4}-\frac{a+c}{2}} \mathcal{I}_{T; \alpha_T, t} \left( l, \frac{k}{\sigma} \right)_{\frac{1}{12}-\frac{\sigma^2}{T^3} \left( \rho^2 + \frac{T^2}{4} - |\rho|T \right), -\frac{1}{12}, \frac{\rho}{T}} \left( c, -\frac{1}{2} - a - c, k; n \right) + E_{r, T}(n). \end{aligned}$$

Here,  $E_{r, T}(n)$  is an error term. If  $r = 2$ , then the error term has the magnitude  $O_T(n \log n)$ , whereas it is of order  $O_{r, T}(n^{r-1})$  if  $r > 2$ .

Theorem A is a direct generalization of [BMR12]. Their theorem states that

$$m_T^r(n) = 2\pi \sum_{k \leq \sqrt{n}} \frac{K_k(n)}{k} \sum_{2a+2b+2c=r} \kappa(a, b, c) (kT)^a (24n-1)^{-\frac{3}{4}+\frac{a}{2}+c} I_{-\frac{3}{2}+a+2c} \left( \frac{\pi\sqrt{24n-1}}{6k} \right) + O(n^{r-1})$$

for  $T = 1$  and  $T = 3$ . Indeed, the additional summand in Theorem A will only occur for  $T > 3$  and comes from the fact that we cannot neglect the term coming from the obstruction to modularity.

In the statement of Theorem A we observe the occurrence of modified Bessel functions with different indices. This accounts to the fact that the moment generating functions  $M_T^r(q)$  exhibit a transformation behavior with mixed weights (in fact the weights range from  $-\frac{1}{2}$  to  $r - \frac{1}{2}$ ). Indeed, the error terms of Theorem A are best possible which one can obtain with the Circle Method. This occurs because the Circle Method cannot detect holomorphic modular forms of positive weight. The size of the error terms in Theorem A are just as big as the coefficients of holomorphic Eisenstein series of the weights occurring in the transformation behavior of the moment generating functions.

From Theorem A we deduce our second main theorem and an analogue of Garvan's conjecture.

<sup>1</sup>We do not give formal definitions for Jacobi forms or mock Jacobi forms, as we will need no results from a general theory, but only require the transformation laws. For an account on the theory of Jacobi forms we refer to [EZ85]. Mock Jacobi forms are a relatively new object of study. For further information the reader may consult [BR10], [Zwe02], and [Zag10].

**Theorem B.** *Let  $T < 24$  be an odd integer and  $r$  an even integer. Then, as  $n \rightarrow \infty$ , we have*

$$m_T^r(n) \sim 2\sqrt{3}(-1)^{\frac{r}{2}} B_r\left(\frac{1}{2}\right) (24n)^{\frac{r}{2}-1} e^{\pi\sqrt{\frac{2n}{3}}},$$

where  $B_r(\cdot)$  is a Bernoulli polynomial. Furthermore,

$$m_{T-2}^r(n) - m_T^r(n) \sim \sqrt{3} \frac{r!}{(r-2)!} (-1)^{\frac{r}{2}+1} B_{r-2}\left(\frac{1}{2}\right) (24n)^{\frac{r}{2}-\frac{3}{2}} e^{\pi\sqrt{\frac{2n}{3}}}.$$

In particular,  $m_{T-2}^r(n) > m_T^r(n)$  for all sufficiently large  $n$ .

## 2. PRELIMINARIES AND NOTATION

We first recall transformation formulas of the  $\eta$ - and  $\theta$ -function and provide transformation laws for Appell-Lerch sums and their completions found by Zwegers [Zwe02]. Furthermore, we set up the notation for the rest of this paper.

**2.1. Transformation laws.** Throughout this work,  $z$  denotes a complex number satisfying  $\operatorname{Re}(z) > 0$  and we set  $q := e^{-2\pi z}$ .

We define the Dedekind  $\eta$ -function by

$$\eta(iz) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n).$$

If  $h, k$  are coprime positive integers then

$$\eta\left(\frac{1}{k}(h + iz)\right) = \sqrt{\frac{i}{z}} \chi(h, [-h]_k, k) \eta\left(\frac{1}{k}([-h]_k + \frac{i}{z})\right),$$

where

$$\chi(h, [-h]_k, k) := \begin{cases} \left(\frac{h}{k}\right) e^{-\frac{\pi i k}{4}} e^{\frac{\pi i}{12}(-\beta[-h]_k(1-k^2)+k(h-[-h]_k))} & \text{if } k \text{ is odd,} \\ e^{-\frac{\pi i}{4}} \left(\frac{k}{h}\right) e^{\frac{\pi i}{12}(hk(1-[-h]_k^2)-[-h]_k(\beta-k+3))} & \text{if } h \text{ is odd.} \end{cases}$$

Here and in the following,  $[\cdot]_k$  denotes the inverse modulo  $k$  and  $\beta$  is defined by  $-h[-h]_k - \beta k = 1$ .

Next, we consider the classical Jacobi theta function, which, for  $v \in \mathbb{C}$ , is defined by

$$\theta(v; iz) := \sum_{\nu \in \frac{1}{2} + \mathbb{Z}} e^{-\pi \nu^2 z + 2\pi i \nu(v + \frac{1}{2})}.$$

For  $\theta(v; iz)$  there is a well-known product expansion:

$$(1) \quad \theta(v; iz) = -ie^{-\frac{\pi z}{4}} e^{-\pi i v} \prod_{n=1}^{\infty} (1 - e^{-2\pi n z}) \left(1 - e^{2\pi i v} e^{-2\pi(n-1)z}\right) (1 - e^{-2\pi i v} e^{-2\pi n z}).$$

The Jacobi theta function satisfies elliptic and modular transformation properties. To state these precisely, let  $h, k$  be coprime integers. Then, for any  $n \in \mathbb{Z}$ , we have

$$\theta(v + 1; iz) = -\theta(v; iz), \quad \theta(-v; iz) = -\theta(v; iz), \quad \theta(v + niz; iz) = (-1)^n e^{\pi n^2 z - 2\pi i n v} \theta(v; iz).$$

Furthermore, we have

$$\theta\left(v; \frac{1}{k}(h + iz)\right) = \sqrt{\frac{i}{z}} \chi^3(h, [-h]_k, k) e^{-\frac{\pi k v^2}{z}} \theta\left(\frac{i v}{z}; \frac{1}{k}([-h]_k + \frac{i}{z})\right).$$

Following Zwegers [Zwe02], for  $u, v \in \mathbb{C}$  and  $u \notin \mathbb{Z} + iz\mathbb{Z}$ , we define the function

$$(2) \quad A(u, v; iz) := e^{\pi i u} \sum_{n \in \mathbb{Z}} \frac{(-1)^n e^{-\pi(n^2+n)z + 2\pi i n v}}{1 - e^{-2\pi n z + 2\pi i u}}.$$

and for  $u, v \in \mathbb{C} \setminus (\mathbb{Z} + iz\mathbb{Z})$ , we define

$$(3) \quad \mu(u, v; iz) := \frac{A(u, v; iz)}{\theta(v; iz)} = \frac{e^{\pi i u}}{\theta(v; iz)} \sum_{n \in \mathbb{Z}} \frac{(-1)^n e^{-\pi(n^2+n)z+2\pi i n v}}{1 - e^{-2\pi n z+2\pi i u}}.$$

The  $\mu$ -function itself does not transform as a modular form. Zwegers discovered that one can complete  $\mu$  to a function  $\hat{\mu}$  having nice transformation properties. To define this completion requires the following non-holomorphic function:

$$(4) \quad R(w; iz) := \sum_{\nu \in \mathbb{Z} + \frac{1}{2}} (-1)^{\nu - \frac{1}{2}} \left( \operatorname{sgn}(\nu) - E\left(\sqrt{2\operatorname{Im}(iz)}\left(\nu + \frac{\operatorname{Im}(w)}{\operatorname{Im}(iz)}\right)\right) \right) e^{\pi \nu^2 z} e^{-2\pi i \nu w},$$

where  $E(z) := 2 \int_0^z e^{-\pi u^2} du$ . Following Zwegers, we define

$$(5) \quad \hat{\mu}(u, v; iz) := \mu(u, v; iz) + \frac{i}{2} R(u - v; iz).$$

**Lemma 2.1** ([Zwe02] Theorem 1.11). *Let  $u, v \in \mathbb{C} \setminus (\mathbb{Z} + iz\mathbb{Z})$ . If  $h, k$  are coprime integers, and  $m, m', n, n' \in \mathbb{Z}$ , then we have:*

- a)  $\hat{\mu}(u + miz + n, v + m'iz + n'; iz) = (-1)^{m+n+m'+n'} e^{-\pi z(m-m')^2+2\pi i(m-m')(u-v)} \hat{\mu}(u, v; iz),$
- b)  $\hat{\mu}(-iuz, -ivz; \frac{1}{k}(h + iz)) = \chi^{-3}(h, [-h]_k, k) \sqrt{\frac{i}{z}} e^{-\pi k z(u-v)^2} \hat{\mu}\left(u, v; \frac{1}{k}([-h]_k + \frac{i}{z})\right).$

The function  $R$  itself also satisfies some transformation formulas. In order to be able to state these, we introduce the Mordell integral for  $w \in \mathbb{C}$  as

$$(6) \quad H(w; z) := \int_{\mathbb{R}} \frac{e^{-\pi i x^2 z - 2\pi w x}}{\cosh \pi x} dx.$$

**Lemma 2.2** ([Zwe02] Propositions 1.9 and 1.10). *For any  $w \in \mathbb{C}$  we have*

- a)  $R(w + 1; iz) = -R(w; iz),$
- b)  $R(w; iz + 1) = e^{-\frac{\pi i}{4}} R(w; iz),$
- c)  $R(w; iz) = -\frac{1}{\sqrt{z}} e^{\frac{\pi w^2}{z}} \left( R\left(\frac{iw}{z}; \frac{i}{z}\right) - H\left(\frac{iw}{z}; \frac{i}{z}\right) \right).$

Furthermore, in [BF11] the following dissection property is proved:

$$d) \quad R\left(w; \frac{iz}{n}\right) = \sum_{l=0}^{n-1} e^{\frac{\pi}{n}(l - \frac{n-1}{2})^2 z} e^{-2\pi i(l - \frac{n-1}{2})(w + \frac{1}{2})} R\left(nw + (l - \frac{n-1}{2})iz + \frac{n-1}{2}; niz\right).$$

Finally, in recent unpublished work [Zwe10], for  $u, v \in \mathbb{C}$  and  $u \notin iz\mathbb{Z} + \mathbb{Z}$  Zwegers introduced a new function  $A_T$  by

$$(7) \quad A_T(u, v; iz) := e^{\pi i u T} \sum_{n \in \mathbb{Z}} \frac{(-1)^{Tn} e^{-\pi T n(n+1)z} e^{2\pi i n v}}{1 - e^{2\pi i u} e^{-2\pi z}}.$$

As observed by Zwegers, this function is related to the functions discussed before in the following way:

**Lemma 2.3.** *The function  $A_T$  satisfies the following properties:*

- a)  $A_T(u, v; iz) = \sum_{t=0}^{T-1} e^{2\pi i u t} A_1\left(Tu, v + tiz + \frac{T-1}{2}; Tiz\right).$
- b)  $A_1(u, v; iz) = \theta(v; iz) \mu(u, v; iz) = A(u, v; iz).$

**2.2. Notation.** We will now introduce the notation that we use for the rest of this paper. Since we (eventually) want to apply the Circle Method, we will be interested in transformations where we replace  $q = e^{-2\pi z}$  by  $e^{\frac{2\pi i}{k}(h+iz)}$ , for  $z \in \mathbb{C}$  with  $\operatorname{Re}(z) > 0$ , for  $k$  being a positive integer, and  $0 \leq h < k$  with  $h$  coprime to  $k$ . We will also assume that  $|z| < 1$ . Instead of repeating these conditions, we will briefly write that “ $h, k$  and  $z$  satisfy the usual conditions”.

For a fixed odd integer  $T$ , we write  $\sigma := (T, k)$  and  $\gamma := \frac{T}{(T, k)}$ . We denote by  $\rho$  the function  $\rho : \mathbb{Z} \rightarrow \{-\frac{T-1}{2}, \dots, \frac{T-1}{2}\}$  which assigns to every integer its residue class modulo  $T$  with smallest absolute value.

In the course of proving transformation formulas, we will encounter several expressions which are roots of unity, and which occur as a result of the roots of unity appearing in the transformation laws of  $\eta, \theta$  and  $\hat{\mu}$ . For odd integers  $T$ ,  $-\frac{T-1}{2} \leq t \leq \frac{T-1}{2}$  and  $h, k$  coprime, we set

$$U_\theta(T, t, h, k) := (-1)^{\frac{(th\gamma - \rho(t\gamma h))}{T}} e^{\frac{\pi i(t\gamma h - \rho(t\gamma h))^2}{\gamma T k}} [-\gamma h]_{\frac{k}{\sigma}} e^{-\frac{2\pi i t \rho(t\gamma h)}{\gamma T k}},$$

$$U_\mu(T, t, h, k) := \chi^{-3}(h, [-h]_k, k) (-1)^{th - \rho(th)} e^{-\frac{\pi i [-h]_k}{k} \left(\frac{th - \rho(th)}{T}\right)^2} e^{\frac{2\pi i t \rho(th)}{T^2 k}},$$

and define  $U_\theta^*(T, t, h, k)$  by

$$\begin{cases} -2\chi^3\left(\gamma h, [-\gamma h]_{\frac{k}{\sigma}}, \frac{k}{\sigma}\right) U_\theta(T, t, h, k) e^{\frac{\pi i}{4k}\sigma[-\gamma h]_{\frac{k}{\sigma}}} \sin\left(-\frac{\pi t}{\gamma k}\left(1 + \gamma h[-\gamma h]_{\frac{k}{\sigma}}\right)\right) & \text{if } \rho(\gamma ht) = 0, \\ -i\chi^3\left(\gamma h, [-\gamma h]_{\frac{k}{\sigma}}, \frac{k}{\sigma}\right) U_\theta(T, t, h, k) e^{\pi i\left(\frac{\sigma}{4k}([- \gamma h]_{\frac{k}{\sigma}} - \left(\frac{\rho(t\gamma h)}{\gamma k}[-\gamma h]_{\frac{k}{\sigma}} - \frac{t}{\gamma k}(1 + \gamma h[-\gamma h]_{\frac{k}{\sigma}})\right)\right)} & \text{if } \rho(\gamma ht) > 0, \\ i\chi^3\left(\gamma h, [-\gamma h]_{\frac{k}{\sigma}}, \frac{k}{\sigma}\right) U_\theta(T, t, h, k) e^{\pi i\left(\frac{\sigma}{4k}([- \gamma h]_{\frac{k}{\sigma}} + \left(\frac{\rho(t\gamma h)}{\gamma k}[-\gamma h]_{\frac{k}{\sigma}} - \frac{t}{\gamma k}(1 + \gamma h[-\gamma h]_{\frac{k}{\sigma}})\right)\right)} & \text{if } \rho(\gamma ht) < 0. \end{cases}$$

Furthermore, for  $0 \leq l \leq k-1$  we set

$$\alpha_{T,t}(l, k) := \frac{1}{k} \left(-\frac{t}{T} + \left(l - \frac{k-1}{2}\right)\right).$$

Note that  $|\alpha_{T,t}(l, k)| < \frac{1}{2}$ . Moreover, we define

$$U_H(T, t, l, h, k) := e^{-\frac{\pi i(hk+1)}{4}} (-1)^{lh + \frac{(k-1)(h-1)}{2} + th - \rho(th) + 1} e^{-\frac{\pi i h}{k} \left(l - \frac{k-1}{2}\right)^2} e^{-2\pi i \left(\left(l - \frac{k-1}{2}\right)\left(\frac{1}{2} - \frac{th}{T k}\right) + \frac{\rho(th)}{T} \alpha_{T,t}(l, k)\right)},$$

and

$$(8) \quad U_H^*(T, t, l, h, k) := i^{\frac{3}{2}} \frac{U_\theta^*(T, t, h, k)}{\chi(h, [-h]_k, k)} U_H\left(T, t, l, \gamma h, \frac{k\gamma}{T}\right) e^{2\pi i \frac{\rho(t\gamma h)}{T} \alpha_{T,t}(l, \frac{k\gamma}{T})} e^{\frac{\pi i h}{12k}} e^{\frac{\pi i [-h]_k}{12k}},$$

Finally, we require the following constants

$$(9) \quad \kappa(a, b, c) := \frac{(2(a+b+c))!}{a!(2b+1)!(2c)!} \frac{(-1)^{a+c}}{\pi^a 2^{2(a+b)}} B_{2c}\left(\frac{1}{2}\right),$$

for  $a, b, c \in \mathbb{N}_0$  and  $\kappa(a, b, c) = 0$  otherwise. Here  $B_{2c}$  denotes the  $2c$ -th Bernoulli polynomial.

$$(10) \quad \kappa^*(a, b, c) := \frac{(2a + (2b+1) + c)!}{a!(2b+1)!c!} \frac{(-1)^{a+c+1}}{\pi^a 2^{2a+2b+1}}$$

for  $a, b, c \in \mathbb{N}_0$  and  $\kappa^*(a, b, c) = 0$  otherwise.

One easily shows that these numbers appear as Taylor coefficients in the Taylor expansions

$$e^{\frac{\pi \nu u^2}{z}} \frac{\sin(\pi u)}{\sinh(\frac{\pi u}{z})} = \sum_{r=0}^{\infty} \sum_{2a+2b+2c=r} \kappa(a, b, c) \nu^a z^{1-a-2c} \frac{(2\pi i u)^r}{r!}$$

and

$$\sin(\pi u) e^{\frac{\pi \nu u^2}{z}} e^{-\frac{2\pi i \lambda u}{z}} = \sum_{r=0}^{\infty} i \sum_{2a+(2b+1)+c=r} \kappa^*(a, b, c) z^{-a-c} \nu^b \lambda^c \frac{(2\pi i u)^r}{r!},$$

### 3. RELATION OF MOMENTS TO TAYLOR COEFFICIENTS OF AN APPELL-LERCH SUM

We now introduce a two-variable generating function for the numbers  $N_T(m, n)$ :

$$\widetilde{\mathcal{M}}_T(x, q) := \sum_{n=0}^{\infty} \sum_{m \in \mathbb{Z}} N_T(m, n) x^m q^n.$$

In [Gar94], Garvan has given several descriptions involving this generating function. In fact, the formula in the next lemma follows easily from three expressions for  $\widetilde{\mathcal{M}}_T(x, q)$  given by Garvan in formula 4.3 and 4.5 of [Gar94].

**Lemma 3.1.** *We have*

$$\widetilde{\mathcal{M}}_T(x, q) = \frac{1-x}{(q)_{\infty}} \sum_{n \in \mathbb{Z}} (-1)^n \frac{q^{\frac{n}{2}(Tn+1)}}{1-xq^n} - \frac{1}{(q)_{\infty}} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{n}{2}(Tn-1)}.$$

A priori, this result holds as a statement about formal power series. To see the connection with the moment generating functions  $M_T^r(q)$ , we now no longer view  $x$  as a formal variable but set  $x := e^{2\pi i u}$  for  $u \in \mathbb{C}$ . Then we find the following Taylor expansion

$$\widetilde{\mathcal{M}}_T(e^{2\pi i u}, q) = \sum_{n=0}^{\infty} \sum_{m \in \mathbb{Z}} N_k(m, n) \sum_{r=0}^{\infty} \frac{(2\pi i u m)^r}{r!} q^n = \sum_{r=0}^{\infty} M_T^r(q) \frac{(2\pi i u)^r}{r!}.$$

Looking at Lemma 3.1, we see that in the expression for  $\widetilde{\mathcal{M}}_T(x, q)$ , the  $x$ -variable only occurs in the first summand and not in the second one. Thus except from the 0-th moment, the second summand does not affect the moment-generating functions for higher moments. As we are interested only in the higher moments<sup>2</sup> we will can equivalently work with the function  $\mathcal{M}_T$  defined by

$$(11) \quad \mathcal{M}_T(u, q) := \frac{1 - e^{2\pi i u}}{(q)_{\infty}} \sum_{n \in \mathbb{Z}} (-1)^n \frac{q^{\frac{n}{2}(Tn+1)}}{1 - e^{2\pi i u} q^n}$$

for  $u, q \in \mathbb{C}$  with  $|q| < 1$ . Then, summarizing the discussion above, we conclude the following:

**Proposition 3.2.** *Let  $r > 0$ . In the Taylor expansion of  $\mathcal{M}_T(u, q)$  at  $u = 0$  the coefficient of  $\frac{(2\pi i u)^r}{r!}$  is equal to the  $r$ -th moment generating function*

$$M_T^r(q) = \sum_{n=0}^{\infty} m_T^r(n) q^n.$$

*Proof.* In fact our reasoning above is only valid if the representation (11) converges. Indeed, it does not converge for all  $u \in \mathbb{C}$ . However, it suffices to know that  $\mathcal{M}_T(u, q)$  is defined in a neighborhood of  $u = 0$ . This is immediate from the following relation between  $\mathcal{M}_T$  and Zwegers' function  $A_T$ .

$$\mathcal{M}_T(u, q) = \frac{(1 - e^{2\pi i u}) q^{\frac{1}{24}}}{\eta(iz)} e^{-\pi i u T} A_T\left(u, -\frac{T-1}{2} iz; iz\right).$$

□

<sup>2</sup>The behavior for the 0-th moment can be treated with classical methods, as Garvan [Gar94] shows that it is the number of partitions where the parts satisfy certain congruence conditions.

It will turn out to be easier not to work with  $\mathcal{M}_T(u, q)$  directly, but to replace the higher level Appell-Lerch sum  $A_T$  by a sum of simpler Appell-Lerch sums using Lemma 2.3. We have

$$\mathcal{M}_T(u, q) = \sum_{t=-\frac{T-1}{2}}^{\frac{T-1}{2}} \mathcal{C}_{T,t}(u, q),$$

where

$$\mathcal{C}_{T,t}(u, q) := -\frac{2i \sin(\pi u) q^{\frac{1}{24}}}{\eta(iz)} e^{2\pi i u t} \mathbf{A}(Tu, tiz; Tiz).$$

We express the functions  $\mathcal{C}_{T,t}$  in yet another way, which already indicates that the behavior is rather different depending on whether  $t = 0$  or  $t \neq 0$ .

**Proposition 3.3.** *We have*

$$\mathcal{C}_{T,t}(u, q) = -\frac{2i \sin(\pi u) q^{\frac{1}{24}}}{\eta(iz)} e^{2\pi i u t} \theta(tiz; Tiz) \mu(Tu, tiz; Tiz) \quad \text{and} \quad \mathcal{C}_{T,0}(u, q) = -\frac{2 \sin(\pi u) q^{\frac{1}{24}} \eta^3(Tiz)}{\eta(iz) \theta(Tu; Tiz)}.$$

*Proof.* The first statement is clear by the definition of  $\mu$  (see equation (3) in the preliminaries). The second statement can be deduced from the following identity in Ramanujans Lost Notebook (see on page 264 of [AB05])

$$\prod_{n=1}^{\infty} \frac{(1 - q^n)^2}{(1 - e^{2\pi i u} q^n)(1 - e^{-2\pi i u} q^n)} = (1 - e^{2\pi i u}) \sum_{n \in \mathbb{Z}} (-1)^n \frac{q^{n(n+1)/2}}{1 - e^{2\pi i u} q^n},$$

and the product formula of the Jacobi theta function.  $\square$

#### 4. TRANSFORMATION LAWS FOR THE FUNCTIONS $\mathcal{C}_{T,t}$

Our next task is to work out transformation laws for  $\mathcal{C}_{T,t}$ . Our treatment here parallels the approach in [BMR12]. In our case there does not occur any new difficulty, only the exposition gets more involved. For that reason we omit a full proof and only briefly indicate how one obtains the transformation laws. A detailed analysis is given in the authors PhD thesis [Wal12].

For  $t = 0$  we deduce the transformation law for  $\mathcal{C}_{T,t}$  directly from those for  $\eta$  and  $\theta$ .

**Proposition 4.1.** *Let  $T > 0$  be an odd integer and suppose that  $h, k$ , and  $z$  satisfy the usual conditions. Then, we have*

$$\mathcal{C}_{T,0}\left(u, e^{\frac{2\pi i}{k}(h+iz)}\right) = -\frac{2}{\gamma} \sqrt{\frac{i}{z}} \frac{\sin(\pi u) e^{\frac{\pi k T u^2}{z}} e^{\frac{\pi i}{12k}(h+iz)}}{\chi(h, [-h]_k, k) \eta\left(\frac{1}{k}([-h]_k + \frac{i}{z})\right)} \frac{\eta^3\left(\frac{T}{\gamma k}([- \gamma h]_{\frac{k}{\sigma}} + \frac{i}{\gamma z})\right)}{\theta\left(\frac{i u T}{\gamma z}; \frac{T}{\gamma k}([- \gamma h]_{\frac{k}{\sigma}} + \frac{i}{\gamma z})\right)}.$$

For  $t \neq 0$  the transformation behavior of

$$\mathcal{C}_{T,t}(u, q) = -\frac{2i \sin(\pi u) q^{\frac{1}{24}}}{\eta(iz)} e^{2\pi i u t} \theta(tiz; Tiz) \mu(Tu, tiz; Tiz)$$

is no longer that of a Jacobi form but rather of a mock Jacobi form because of the appearance of the  $\mu$ -function. As the transformation law of the theta function is well-known we only have to find that of the  $\mu$ -function.

To determine the transformation law for the  $\mu$ -function we argue as in [BMR12]. First we write

$$(12) \quad \mu\left(u, \frac{t}{T k}(h+iz); \frac{1}{k}(h+iz)\right) = \widehat{\mu}\left(u, \frac{t}{T k}(h+iz); \frac{1}{k}(h+iz)\right) - \frac{i}{2} R\left(u - \frac{t}{T k}(h+iz); \frac{1}{k}(h+iz)\right).$$



To the functions on the right hand side of (12), we can apply the transformation properties as stated in the preliminaries. However, in the transformation laws for both summands there will also occur non-holomorphic terms involving the  $R$ -function. On the other hand, we know that  $\mu$  is a holomorphic function. This implies that the non-holomorphic terms appearing in the transformation laws of both summands in (12) have to cancel. The main technical issue is now to identify these non-holomorphic terms and prove that they indeed cancel. To achieve this, one has make further use of the transformation laws for  $R$  and  $\hat{\mu}$  in order to obtain non-holomorphic parts of a similar shape. Indeed, one proves that

$$R\left(u - \frac{t}{Tk}(h + iz); \frac{1}{k}(h + iz)\right) = \sqrt{\frac{i}{kz}} e^{-\frac{\pi t^2 z}{T^2 k} + \frac{\pi k}{z} \left(u - \frac{\rho(th)}{Tk}\right)^2 - \frac{2\pi i u t}{T}} \\ \times \sum_{l=0}^{k-1} U_H(T, t, l, h, k) \left( R\left(\frac{i u}{z} - \frac{\rho(th)i}{Tkz} - \alpha_{T,t}(l, k); \frac{i}{kz}\right) - H\left(\frac{i u}{z} - \frac{\rho(th)i}{Tkz} - \alpha_{T,t}(l, k); \frac{i}{kz}\right) \right).$$

and

$$\hat{\mu}\left(u, \frac{t}{Tk}(h + iz); \frac{1}{k}(h + iz)\right) = \sqrt{\frac{i}{z}} e^{\frac{\pi k}{z} \left(u - \frac{\rho(th)}{Tk}\right)^2 - \frac{t^2 \pi z}{T^2 k} - \frac{2\pi i u t}{T}} \\ \times U_\mu(T, t, h, k) \hat{\mu}\left(\frac{i u}{z}, \frac{\rho(th)}{Tk} ([ -h ]_k + \frac{i}{z}) - \frac{t}{Tk} (1 + h [ -h ]_k); \frac{1}{k} ([ -h ]_k + \frac{i}{z})\right).$$

Now, in both equations there occur non-holomorphic parts coming from the  $R$ -function (in the second equation these come from the definition of  $\hat{\mu}$ ). Now we can employ the same argument as in [BMR12] involving the Fourier-Whittaker expansion of the  $R$ -functions in order to prove that these non-holomorphic terms cancel in (12). Working this out, we obtain a transformation formula for  $\mu\left(u, \frac{t}{T}iz; iz\right)$  and use it to find the following transformation law for  $\mathcal{C}_{T,t}$ .

**Proposition 4.2.** *Let  $T > 0$  be an odd integer and suppose that  $h, k$ , and  $z$  satisfy the usual conditions. For  $t \neq 0$ , we have*

$$\mathcal{C}_{T,t}\left(u, e^{\frac{2\pi i}{k}(h+iz)}\right) = -\frac{2i \sin(\pi u) e^{\frac{\pi i}{12k}(h+iz)}}{\eta\left(\frac{1}{k}(h+iz)\right)} \theta\left(\frac{t}{k}(h+iz); \frac{T}{k}(h+iz)\right) \sqrt{\frac{i}{\gamma z}} e^{\frac{\pi k}{T^2 z} \left(Tu - \frac{\rho(t\gamma h)}{\gamma k}\right)^2 - \frac{t^2 \pi z}{T^2 k}} \\ \left( U_\mu\left(T, t, \gamma h, \frac{k}{\sigma}\right) \mu\left(\frac{i u T}{\gamma z}, \frac{\rho(t\gamma h)}{\gamma k} ([ -\gamma h ]_{\frac{k}{\sigma}} + \frac{i}{\gamma z}) - \frac{t}{\gamma k} (1 + \gamma h [ -\gamma h ]_{\frac{k}{\sigma}}); \frac{\sigma}{k} ([ -\gamma h ]_{\frac{k}{\sigma}} + \frac{i}{\gamma z})\right) \right. \\ \left. + \frac{i}{2\sqrt{\frac{k\gamma}{T}}} \sum_{l=0}^{\frac{k\gamma}{T}-1} U_H\left(T, t, l, \gamma h, \frac{k\gamma}{T}\right) H\left(\frac{i u T}{\gamma z} - \frac{\rho(t\gamma h)i}{\gamma^2 k z} - \alpha_{T,t}(l, \frac{k\gamma}{T}); \frac{T i}{\gamma^2 k z}\right) \right).$$

## 5. ASYMPTOTIC EXPANSIONS FOR THE MOMENT GENERATING FUNCTIONS $M_T^r(q)$

By Proposition 3.2 we know that, for  $r > 0$ , the  $r$ -th Taylor coefficient in the Taylor expansion of

$$(13) \quad \mathcal{M}_T\left(u, e^{\frac{2\pi i}{k}(h+iz)}\right) = \sum_{t=-\frac{T-1}{2}}^{\frac{T-1}{2}} \mathcal{C}_{T,t}\left(u, e^{\frac{2\pi i}{k}(h+iz)}\right)$$

at  $u = 0$  is equal to  $M_T^r \left( e^{\frac{2\pi i}{k}(h+iz)} \right)$ . If we now apply the transformation properties of Proposition 4.1 and Proposition 4.2 to the functions appearing on the right hand side of (13) we obtain

$$(14) \quad \mathcal{M}_T \left( u, e^{\frac{2\pi i}{k}(h+iz)} \right) = \mathcal{C}_{T,0} \left( u, e^{\frac{2\pi i}{k}(h+iz)} \right) + \sum_{\substack{t=-\frac{T-1}{2} \\ t \neq 0}}^{\frac{T-1}{2}} \left( \mathcal{C}_{T,t}^\mu \left( u, e^{\frac{2\pi i}{k}(h+iz)} \right) + \mathcal{C}_{T,t}^H \left( u, e^{\frac{2\pi i}{k}(h+iz)} \right) \right),$$

where  $\mathcal{C}_{T,0}$ ,  $\mathcal{C}_{T,t}^\mu$  and  $\mathcal{C}_{T,t}^H$  are defined by

$$(15) \quad \mathcal{C}_{T,0} \left( u, e^{\frac{2\pi i}{k}(h+iz)} \right) := -\frac{2}{\gamma} \sqrt{\frac{i}{z}} e^{\frac{\pi k u^2 T}{z}} \frac{\sin(\pi u) e^{\frac{\pi i}{12k}(h+iz)}}{\chi(h, [-h]_k, k) \eta\left(\frac{1}{k}([-h]_k + \frac{i}{z})\right)} \frac{\eta^3\left(\frac{T}{\gamma k} \left([- \gamma h]_{\frac{k}{\sigma}} + \frac{i}{\gamma z}\right)\right)}{\theta\left(\frac{i u T}{\gamma z}; \frac{T}{\gamma k} \left([- \gamma h]_{\frac{k}{\sigma}} + \frac{i}{\gamma z}\right)\right)},$$

$$(16) \quad \begin{aligned} \mathcal{C}_{T,t}^\mu \left( u, e^{\frac{2\pi i}{k}(h+iz)} \right) &:= -\sqrt{\frac{i}{\gamma z}} \frac{2i \sin(\pi u) e^{\frac{\pi i}{12k}(h+iz)}}{\eta\left(\frac{1}{k}(h+iz)\right)} \theta\left(\frac{t}{k}(h+iz); \frac{T}{k}(h+iz)\right) e^{\frac{\pi k}{Tz} \left(Tu - \frac{\rho(t\gamma h)}{\gamma k}\right)^2 - \frac{t^2 \pi z}{Tk}} \\ &\quad \times U_\mu\left(T, t, \gamma h, \frac{k}{\sigma}\right) \mu\left(\frac{i u T}{\gamma z}, \frac{\rho(t\gamma h)}{\gamma k} \left([- \gamma h]_{\frac{k}{\sigma}} + \frac{i}{\gamma z}\right) - \frac{t}{\gamma k} \left(1 + \gamma h [- \gamma h]_{\frac{k}{\sigma}}\right); \frac{\sigma}{k} \left([- \gamma h]_{\frac{k}{\sigma}} + \frac{i}{\gamma z}\right)\right), \end{aligned}$$

and

$$(17) \quad \begin{aligned} \mathcal{C}_{T,t}^H \left( u, e^{\frac{2\pi i}{k}(h+iz)} \right) &:= \frac{1}{\gamma \sqrt{\frac{k}{T}}} \sqrt{\frac{i}{z}} \frac{\sin(\pi u) e^{\frac{\pi i}{12k}(h+iz)}}{\eta\left(\frac{1}{k}(h+iz)\right)} \theta\left(\frac{t}{k}(h+iz); \frac{T}{k}(h+iz)\right) e^{\frac{\pi k}{Tz} \left(Tu - \frac{\rho(t\gamma h)}{\gamma k}\right)^2 - \frac{t^2 \pi z}{Tk}} \\ &\quad \times \sum_{l=0}^{\frac{k\gamma}{T}-1} U_H\left(T, t, l, \gamma h, \frac{k\gamma}{T}\right) H\left(\frac{i u T}{\gamma z} - \frac{\rho(t\gamma h)i}{\gamma^2 k z} - \alpha_{T,t}\left(l, \frac{k\gamma}{T}\right); \frac{T i}{\gamma^2 k z}\right). \end{aligned}$$

Now, in order to find an expression for  $M_T^r \left( e^{\frac{2\pi i}{k}(h+iz)} \right)$ , we will have to determine Taylor expansions of these three expressions with respect to  $u$ . As this will give very messy formulas and since we are only interested in asymptotic expressions anyways we content ourself with asymptotic Taylor expansions. By this we mean that in the Taylor expansion with respect to  $u$  we split the Taylor coefficients (which will be functions in  $z$ ) into a main part and an error part.

As in [BMR12], the contributions coming from  $\mathcal{C}_{T,0}$  and  $\mathcal{C}_{T,t}^\mu$  can be jointly expressed in a nice way. In [BMR12] there was no need to consider the contribution from  $\mathcal{C}_{T,t}^H$  explicitly, as it was part of the error term. In the case  $T > 3$ , we can no longer neglect these contributions and have to determine an asymptotic Taylor expansion, as well.

**5.1. The contribution from  $\mathcal{C}_{T,0}$  and  $\mathcal{C}_{T,t}^\mu$ .** In order to derive asymptotic Taylor expansions for  $\mathcal{C}_{T,0}$  and  $\mathcal{C}_{T,t}^\mu$ , we have to first find asymptotic Taylor expansions for the  $\theta$ -function and the  $\mu$ -function appearing in (15) and (16), respectively. For that purpose we first use that the  $\mu$ -function is the quotient of A and a  $\theta$ -function. To find asymptotic Taylor expansions  $\theta$  and A we consider their series representations, identify the main parts and bound the other summands into the error term. Working this out is rather straightforward but involves a few lengthy computations. We omit the proof and refer Section 2 of Part B of the author's PhD thesis [Wal12] for a more detailed analysis.

Suppose that  $T > 0$  is an odd integer and suppose  $h, k$ , and  $z$  satisfy the usual conditions. Furthermore, assume that  $\operatorname{Re}(\frac{1}{z}) \geq \frac{k}{2}$ . Then,

$$\theta\left(\frac{i u T}{\gamma z}; \frac{T}{\gamma k} \left([- \gamma h]_{\frac{k}{\sigma}} + \frac{i}{\gamma z}\right)\right)^{-1} = i \frac{e^{-\frac{\pi i T}{4 \gamma k} \left([- \gamma h]_{\frac{k}{\sigma}} + \frac{i}{\gamma z}\right)}}{2 \sinh\left(\frac{T \pi u}{\gamma z}\right)} + \frac{1}{u} \sum_{r=0}^{\infty} c_{r,T,h,k}(z) \frac{(2 \pi i u)^r}{r!},$$

where  $c_{r,T,h,k} : \frac{1}{i} \mathbb{H} \rightarrow \mathbb{C}$  are functions which satisfy  $|c_{r,T,h,k}(z)| \ll_{r,T} |z|^{1-r} e^{-\frac{7 T \pi}{4 \gamma^2 k} \operatorname{Re}(\frac{1}{z})}$ .

Similarly, we may decompose the Appell-Lerch sum

$$(18) \quad A\left(\frac{i u T}{\gamma z}, \frac{\rho(t \gamma h)}{\gamma k} \left([- \gamma h]_{\frac{k}{\sigma}} + \frac{i}{\gamma z}\right) - \frac{t}{\gamma k} \left(1 + \gamma h [- \gamma h]_{\frac{k}{\sigma}}\right); \frac{\sigma}{k} \left([- \gamma h]_{\frac{k}{\sigma}} + \frac{i}{\gamma z}\right)\right).$$

as

$$\frac{1}{2 \sinh\left(\frac{\pi u T}{\gamma z}\right)} + \sum_{r=0}^{\infty} c_{r,T,t,h,k}(z) \frac{(2 \pi i u)^r}{r!}$$

with certain functions  $c_{r,t,h,k,T} : \frac{1}{i} \mathbb{H} \rightarrow \mathbb{C}$  that satisfy  $|c_{r,t,h,k,T}(z)| \ll_{r,T} |z|^{-r} e^{-\frac{2 \pi}{\gamma^2 k} (T - |\rho(t \gamma h)|) \operatorname{Re}(\frac{1}{z})}$ .

In order to write down an asymptotic Taylor expansion for  $\mathcal{C}_{T,0}$  and  $\mathcal{C}_{T,t}^{\mu}$  it remains to combine the asymptotic Taylor expansion above with the asymptotic behavior of the  $\eta$ - and  $\theta$ -functions appearing in (15) and (16). Doing so, we obtain the following two propositions.

**Proposition 5.1.** *Let  $T > 0$  be an odd integer and suppose that  $h, k$ , and  $z$  satisfy the usual conditions. Furthermore, assume that  $\operatorname{Re}(\frac{1}{z}) \geq \frac{k}{2}$ . Then,*

$$\mathcal{C}_{T,0}\left(u, e^{\frac{2 \pi i}{k}(h+i z)}\right) = -\frac{i}{\gamma} \sqrt{\frac{i}{z}} e^{\frac{\pi k u^2 T}{z}} \frac{\sin(\pi u) e^{\frac{\pi i}{12 k}(h+i z)} e^{-\frac{\pi i}{12 k}([-h]_k + \frac{i}{z})}}{\sinh\left(\frac{T \pi u}{\gamma z}\right) \chi(h, [-h]_k, k)} + \sum_{r=0}^{\infty} c_{r,T,0,h,k}(z) \frac{(2 \pi i u)^r}{r!},$$

with functions  $c_{r,T,h,k} : \frac{1}{i} \mathbb{H} \rightarrow \mathbb{C}$  which satisfy

$$|c_{2,T,0,h,k}(z)| \ll_T |z|^{-\frac{1}{2}} e^{-\frac{\pi}{k} \left(\frac{5 T}{2 \gamma^2} - \frac{1}{12}\right) \operatorname{Re}(\frac{1}{z})} \quad \text{and} \quad |c_{r,T,0,h,k}(z)| \ll_{r,T} k^{\frac{r}{2}} |z|^{\frac{1}{2}-r} e^{-\frac{\pi}{k} \left(\frac{5 T}{2 \gamma^2} - \frac{1}{12}\right) \operatorname{Re}(\frac{1}{z})}.$$

**Proposition 5.2.** *Let  $T > 0$  be an odd integer and suppose that  $h, k$ , and  $z$  satisfy the usual conditions. Furthermore, assume that  $\operatorname{Re}(\frac{1}{z}) \geq \frac{k}{2}$ . For any  $t \neq 0$ , we find that*

$$\mathcal{C}_{T,t}^{\mu}\left(u, e^{\frac{2 \pi i}{k}(h+i z)}\right) = -\frac{i}{\gamma} \sqrt{\frac{i}{z}} \frac{\sin(\pi u) e^{\frac{\pi i}{12 k}(h+i z)} e^{-\frac{\pi i}{12 k}([-h]_k + \frac{i}{z})}}{\sinh\left(\frac{T \pi u}{\gamma z}\right) \chi(h, [-h]_k, k)} e^{\frac{\pi k T u^2}{z} - \frac{2 \pi u \rho(t \gamma h)}{\gamma z}} + \sum_{r=0}^{\infty} c_{r,T,t,h,k}(z) \frac{(2 \pi i u)^r}{r!},$$

where  $c_{r,T,t,h,k} : \frac{1}{i} \mathbb{H} \rightarrow \mathbb{C}$  are functions that satisfy

$$|c_{2,T,t,h,k}(z)| \ll_T |z|^{-\frac{3}{2}} e^{-\frac{2 \pi}{k} \operatorname{Re}(\frac{1}{z}) \left(\frac{T - |\rho(t \gamma h)|}{\gamma^2} - \frac{1}{24}\right)}$$

and, for  $r > 2$ ,

$$|c_{r,T,t,h,k}(z)| \ll_{r,T} k^{\frac{r}{2}} |z|^{\frac{1}{2}-r} e^{-\frac{2 \pi}{k} \operatorname{Re}(\frac{1}{z}) \left(\frac{T - |\rho(t \gamma h)|}{\gamma^2} - \frac{1}{24}\right)}.$$

We see that the main terms of the asymptotic expressions in Proposition 5.1 and Proposition 5.2 are very similar, and that it might be possible to combine all contributions in (14) coming from  $\mathcal{C}_{T,0}$  and  $\mathcal{C}_{T,t}^{\mu}$  into one single formula. In fact, this reduces to evaluate the sum

$$\frac{1}{\gamma \sinh\left(\frac{\pi u T}{\gamma z}\right)} \sum_{t=-\frac{T-1}{2}}^{\frac{T-1}{2}} e^{-\frac{2 \pi u \rho(t \gamma h)}{\gamma z}}.$$

Using the definition of the  $\rho$ -function, it is easy to deduce that this expression is always (i.e. independently of  $k$ ) equal to  $\frac{1}{\sinh(\frac{\pi u}{z})}$ . Furthermore, the error terms in Proposition 5.1 and Proposition 5.2 are also of a similar shape. Indeed, using the fact that  $\frac{5T}{4\gamma^2} \geq \frac{1}{T}$  and  $\frac{T-|\rho(t\gamma h)|}{\gamma^2} \geq \frac{1}{T}$ , we easily arrive at the following result.

**Proposition 5.3.** *Let  $T > 0$  be an odd integer and suppose that  $h, k$ , and  $z$  satisfy the usual conditions. Furthermore, assume that  $\operatorname{Re}(\frac{1}{z}) \geq \frac{k}{2}$ . Then, we have*

$$\begin{aligned} \mathcal{C}_{T,0}\left(u, e^{\frac{2\pi i}{k}(h+iz)}\right) &+ \sum_{\substack{t=-\frac{T-1}{2} \\ t \neq 0}}^{\frac{T-1}{2}} \mathcal{C}_{T,t}^\mu\left(u, e^{\frac{2\pi i}{k}(h+iz)}\right) \\ &= -i^{\frac{3}{2}} \frac{1}{\sqrt{z}} \frac{\sin(\pi u)}{\sinh(\frac{\pi u}{z})} e^{\frac{\pi i}{12k}(h-[h]_k)} \chi^{-1}(h, [-h]_k, k) e^{-\frac{\pi}{12k}(1-\frac{1}{z})} e^{\frac{\pi k T u^2}{z}} + \sum_{r=0}^{\infty} c_{r,T,h,k}(z) \frac{(2\pi i u)^r}{r!}, \end{aligned}$$

where  $c_{r,T,h,k} : \frac{1}{i}\mathbb{H} \rightarrow \mathbb{C}$  are functions that satisfy

$$|c_{2,T,h,k}(z)| \ll_T |z|^{-\frac{3}{2}} e^{-\frac{2\pi}{k}\operatorname{Re}(\frac{1}{z})(\frac{1}{T}-\frac{1}{24})} \quad \text{and} \quad |c_{r,T,h,k}(z)| \ll_{r,T} k^{\frac{r}{2}} |z|^{\frac{1}{2}-r} e^{-\frac{2\pi}{k}\operatorname{Re}(\frac{1}{z})(\frac{1}{T}-\frac{1}{24})}.$$

The main term may be rewritten as

$$-i^{\frac{3}{2}} e^{\frac{\pi i}{12k}(h-[h]_k)} \chi^{-1}(h, [-h]_k, k) e^{-\frac{\pi}{12k}(z-\frac{1}{z})} \sum_{r=0}^{\infty} \sum_{2a+2b+2c=r} \kappa(a, b, c)(kT)^a z^{\frac{1}{2}-a-2c} \frac{(2\pi i u)^r}{r!}.$$

**5.2. The contribution from  $\mathcal{C}_{T,t}^H$ .** In order to obtain an asymptotic Taylor expansion of (17) we will rewrite this expression first. Grouping all terms that depend on  $u$ , we can write (17) as follows:

$$\begin{aligned} \mathcal{C}_{T,t}^H\left(u, e^{\frac{2\pi i}{k}(h+iz)}\right) &= \frac{1}{\gamma\sqrt{\frac{k}{T}}} \sqrt{\frac{i}{z}} \frac{e^{\frac{\pi i}{12k}(h+iz)}}{\eta\left(\frac{1}{k}(h+iz)\right)} \theta\left(\frac{t}{k}(h+iz); \frac{T}{k}(h+iz)\right) e^{\frac{\pi\rho(t\gamma h)^2}{\gamma^2 k T z} - \frac{t^2 \pi z}{T k}} \\ (19) \quad &\times \sum_{l=0}^{\frac{k\gamma}{T}-1} U_H\left(T, t, l, \gamma h, \frac{k\gamma}{T}\right) \sin(\pi u) e^{\frac{\pi k T u^2}{z}} e^{-\frac{2\pi u \rho(t\gamma h)}{\gamma z}} H\left(\frac{i u T}{\gamma z} - \frac{\rho(t\gamma h) i}{\gamma^2 k z} - \alpha_{T,t}\left(l, \frac{k\gamma}{T}\right); \frac{T i}{\gamma^2 k z}\right). \end{aligned}$$

We can now find the Taylor expansion with respect to  $u$  by simply moving all the functions depending on  $u$  into the defining integral of the Mordell-integral  $H$ , computing the Taylor expansion inside the integral and interchanging summation and integration (which needs to and can be justified). In order to state our result, we define

$$(20) \quad \mathcal{H}_{c,T}(\alpha, \gamma, \varrho, k; z) := \int_{\mathbb{R}} (x + i\varrho)^c \frac{e^{-\frac{\pi T x^2}{\gamma^2 k z} + 2\pi x \alpha}}{\cosh(\pi(x + i\varrho))} dx.$$

**Lemma 5.4.** *Let  $T > 0$  be an odd integer. Further suppose that  $\alpha, \gamma, \varrho \in \mathbb{Q}$  and that  $k$  and  $z$  satisfy the usual conditions. Then, we have the following Taylor expansion at  $u = 0$ :*

$$\begin{aligned} &\sin(\pi u) e^{\frac{\pi k T u^2}{z}} e^{-\frac{2\pi u \varrho T}{\gamma z}} H\left(\frac{T i u}{\gamma z} - \frac{T i \varrho}{\gamma^2 k z} - \alpha; \frac{T i}{\gamma^2 k z}\right) \\ &= e^{-\frac{\pi T \varrho^2}{\gamma^2 k z} + 2\pi i \varrho \alpha} \sum_{r=0}^{\infty} \sum_{2a+(2b+1)+c=r} \left(i\kappa^*(a, b, c) z^{-a-c} (kT)^b \left(\frac{T}{\gamma}\right)^c \mathcal{H}_{c,T}(\alpha, \gamma, \varrho, k; z)\right) \frac{(2\pi i u)^r}{r!}. \end{aligned}$$

Combining Lemma 5.4 (with parameters  $\gamma = \gamma, \alpha = \alpha_{T,t} \left( l, \frac{k\gamma}{T} \right), \varrho = \frac{\rho(t\gamma h)}{T}$ ) with (19), we obtain that  $\mathcal{C}_{T,t}^H \left( u, e^{\frac{2\pi i}{k}(h+iz)} \right)$  is equal to

$$(21) \quad \frac{1}{\gamma \sqrt{\frac{k}{T}}} \sqrt{\frac{i}{z}} \frac{e^{\frac{\pi i}{12k}(h+iz)}}{\eta \left( \frac{1}{k}(h+iz) \right)} \theta \left( \frac{t}{k}(h+iz); \frac{T}{k}(h+iz) \right) e^{-\frac{t^2 \pi z}{Tk}} \sum_{l=0}^{\frac{k\gamma}{T}-1} U_H \left( T, t, l, \gamma h, \frac{k\gamma}{T} \right) e^{2\pi i \frac{\rho(t\gamma h)}{T} \alpha_{T,t} \left( l, \frac{k\gamma}{T} \right)} \\ \times \sum_{r=0}^{\infty} \sum_{2a+(2b+1)+c=r} \left( i\kappa^*(a, b, c) z^{-a-c} (kT)^b \left( \frac{T}{\gamma} \right)^c \mathcal{H}_{c,T} \left( \alpha_{T,t} \left( l, \frac{k\gamma}{T} \right), \gamma, \frac{\rho(t\gamma h)}{T}, k; z \right) \right) \frac{(2\pi i u)^r}{r!}.$$

Now we can determine the asymptotic Taylor expansion of (17).

**Proposition 5.5.** *Let  $T > 0$  be an odd integer and suppose that  $h, k$ , and  $z$  satisfy the usual conditions. Suppose that  $\operatorname{Re} \left( \frac{1}{z} \right) \geq \frac{k}{2}$ . Let  $c$  and  $r$  be positive integers with  $c \leq r$  and  $t \neq 0$ . Then,*

$$\mathcal{C}_{T,t}^H \left( u, e^{\frac{2\pi i}{k}(h+iz)} \right) = \gamma^{-\frac{3}{2}} \sqrt{\frac{T}{k}} e^{-\frac{\pi z}{12k}} e^{-\frac{\pi}{\gamma^2 T k z} \left( |\rho(t\gamma h)|^2 + \frac{T^2}{4} - |\rho(t\gamma h)|T \right) + \frac{\pi}{12kz}} \sum_{l=0}^{\frac{k\gamma}{T}-1} U_H^*(T, t, l, h, k) \\ \times \sum_{r=0}^{\infty} \sum_{2a+(2b+1)+c=r} \kappa^*(a, b, c) z^{-\frac{1}{2}-a-c} (kT)^b \left( \frac{T}{\gamma} \right)^c \mathcal{H}_{c,T} \left( \alpha_{T,t} \left( l, \frac{k\gamma}{T} \right), \gamma, \frac{\rho(t\gamma h)}{T}, k; z \right) \frac{(2\pi i u)^r}{r!} \\ + \sum_{r=0}^{\infty} c_{r,T,t,h,k}(z) \frac{(2\pi i u)^r}{r!},$$

where  $c_{r,T,t,h,k} : \frac{1}{i}\mathbb{H} \rightarrow \mathbb{C}$  are functions that satisfy

$$|c_{2,T,t,h,k}(z)| \ll_{r,T} k^{\frac{1}{2}} |z|^{-\frac{3}{2}} e^{-\frac{\pi}{k} \left( \frac{9}{4T} - \frac{1}{12} \right) \operatorname{Re} \left( \frac{1}{z} \right)} \quad \text{and} \quad |c_{r,T,t,h,k}(z)| \ll_{r,T} k^{\frac{r}{2}} |z|^{-r+\frac{1}{2}} e^{-\frac{\pi}{k} \left( \frac{9}{4T} - \frac{1}{12} \right) \operatorname{Re} \left( \frac{1}{z} \right)}.$$

*Proof.* First, we investigate the asymptotic behavior of the quotient of the  $\theta$ -function and  $\eta$ -function in (21). In order to do that, we apply the transformation formulas for  $\eta$  and  $\theta$  and then consider the defining series of the  $\eta$ -function and  $\theta$ -function. We extract the leading term and bound the other terms into an error part. This yields that

$$\frac{\theta \left( \frac{t}{k}(h+iz); \frac{T}{k}(h+iz) \right)}{\eta \left( \frac{1}{k}(h+iz) \right)}$$

is equal to

$$\frac{1}{\sqrt{\gamma}} \frac{U_{\theta}^*(T, t, h, k) e^{\frac{\pi i}{12k}[-h]_k}}{\chi(h, [-h]_k, k)} e^{-\frac{\pi}{\gamma^2 T k z} \left( \rho(t\gamma h)^2 + \frac{T^2}{4} - |\rho(t\gamma h)|T \right) + \frac{1}{12kz}} e^{\frac{\pi t^2 z}{Tk}} + O_T \left( e^{-\frac{9\pi}{4Tk} \operatorname{Re} \left( \frac{1}{z} \right) + \frac{\pi}{12k} \operatorname{Re} \left( \frac{1}{z} \right)} \right).$$

Now the main part is easily read off and it remains to bound the error term. Taking absolute values, we see that the error term  $|c_{r,T,t,h,k}(z)|$  is essentially bounded by

$$\left| \frac{1}{\gamma \sqrt{\frac{k}{T}}} \sqrt{\frac{i}{z}} e^{\frac{\pi i}{12k}(h+iz)} e^{-\frac{9\pi}{4Tk} \operatorname{Re} \left( \frac{1}{z} \right) + \frac{\pi}{12k} \operatorname{Re} \left( \frac{1}{z} \right)} e^{\frac{t^2 \pi z}{Tk}} \sum_{l=0}^{\frac{k\gamma}{T}-1} U_H \left( T, t, l, \gamma h, \frac{k\gamma}{T} \right) e^{2\pi i \frac{\rho(t\gamma h)}{T} \alpha_{T,t} \left( l, \frac{k\gamma}{T} \right)} \right. \\ \left. \times \sum_{2a+(2b+1)+c=r} \left( i\kappa^*(a, b, c) z^{-a-c} (kT)^b \left( \frac{T}{\gamma} \right)^c \mathcal{H}_{c,T} \left( \alpha_{T,t} \left( l, \frac{k\gamma}{T} \right), \gamma, \frac{\rho(t\gamma h)}{T}, k; z \right) \right) \right|.$$

Introducing a uniform bound for  $\kappa^*$  in  $r$  and estimating all constants, we see that

$$|c_{r,T,t,h,k}(z)| \ll_{r,T} e^{-\frac{\pi}{k}(\frac{9}{4T}-\frac{1}{12})\text{Re}(\frac{1}{z})} \sum_{2a+(2b+1)+c=r} k^{b-\frac{1}{2}} |z|^{-\frac{1}{2}-a-c} \sum_{l=0}^{\frac{k\gamma}{T}-1} \left| \mathcal{H}_{c,T} \left( \alpha_{T,t} \left( l, \frac{k\gamma}{T} \right), \gamma, \frac{\rho(t\gamma h)}{T}, k; z \right) \right|.$$

Now, we observe that

$$\sum_{l=0}^{\frac{k\gamma}{T}-1} \left| \mathcal{H}_{c,T} \left( \alpha_{T,t} \left( l, \frac{k\gamma}{T} \right), \gamma, \frac{\rho(t\gamma h)}{T}, k; z \right) \right| \leq k \max_{c \leq r, l \leq k} \left| \mathcal{H}_{c,T} \left( \alpha_{T,t} \left( l, \frac{k\gamma}{T} \right), \gamma, \frac{\rho(t\gamma h)}{T}, k; z \right) \right|.$$

Using the fact that  $\left| \alpha_{T,t} \left( l, \frac{k\gamma}{T} \right) \right| < \frac{1}{2}$  and  $\left| \frac{\rho(t\gamma h)}{T} \right| < \frac{1}{2}$ , one shows that

$$\left| \mathcal{H}_{c,T} \left( \alpha_{T,t} \left( l, \frac{k\gamma}{T} \right), \gamma, \frac{\rho(t\gamma h)}{T}, k; z \right) \right| \ll_{c,T} 1 \ll_{r,T} 1.$$

This yields

$$|c_{r,T,t,h,k}(z)| \ll_{r,T} e^{-\frac{\pi}{k}(\frac{9}{4T}-\frac{1}{12})\text{Re}(\frac{1}{z})} \sum_{2a+(2b+1)+c=r} k^{b+\frac{1}{2}} |z|^{-\frac{1}{2}-a-c}.$$

We distinguish two cases. If  $r = 2$ , then we have that  $a = b = 0$  and  $c = 1$ . Then we obtain

$$|c_{2,t,T,h,k}(z)| \ll_{r,T} k^{\frac{1}{2}} |z|^{-\frac{3}{2}} e^{-\frac{\pi}{k}(\frac{9}{4T}-\frac{1}{12})\text{Re}(\frac{1}{z})}.$$

If  $r > 2$ , we bound  $k^b$  by  $k^{\frac{r-1}{2}}$ . We also see that  $|z|^{-a-c} \leq |z|^{-r+1}$ .  $\square$

**5.3. Asymptotic expansion of  $M_T^r$  at roots of unity.** In Proposition 5.3 and in Proposition 5.5 we have found asymptotic Taylor expansions for

$$\mathcal{C}_{T,0} \left( u, e^{\frac{2\pi i}{k}(h+iz)} \right) + \sum_{\substack{t=-\frac{T-1}{2} \\ t \neq 0}}^{\frac{T-1}{2}} \mathcal{C}_{T,t}^\mu \left( u, e^{\frac{2\pi i}{k}(h+iz)} \right) \quad \text{and} \quad \mathcal{C}_{T,t}^H \left( u, e^{\frac{2\pi i}{k}(h+iz)} \right),$$

respectively. Combining these results leads to an asymptotic Taylor expansion for  $\mathcal{M}_T \left( u, e^{\frac{2\pi i}{k}(h+iz)} \right)$ .

Referring to Proposition 3.2, we can then immediately derive asymptotic formulas for  $M_T^r \left( e^{\frac{2\pi i}{k}(h+iz)} \right)$ . To state our result, we first define the functions

$$(22) \quad M_T^{r,\mu}(h, k; z) := -i^{\frac{3}{2}} e^{\frac{\pi i}{12k}(h-[h]_k)} \chi^{-1}(h, [-h]_k, k) e^{-\frac{\pi}{12k}(z-\frac{1}{z})} \sum_{2a+2b+2c=r} \kappa(a, b, c) (kT)^a z^{\frac{1}{2}-a-2c}$$

and

(23)

$$\begin{aligned} M_T^{r,H}(t, l, h, k; z) := & \gamma^{-\frac{3}{2}} \sqrt{\frac{T}{k}} e^{-\frac{\pi z}{12k}} e^{-\frac{\pi}{\gamma^2 T k z} \left( |\rho(t\gamma h)|^2 + \frac{T^2}{4} - |\rho(t\gamma h)|T \right) + \frac{\pi}{12kz}} \sum_{l=0}^{\frac{k\gamma}{T}-1} U_H^*(T, t, l, h, k) \\ & \times \sum_{2a+(2b+1)+c=r} \kappa^*(a, b, c) z^{-\frac{1}{2}-a-c} (kT)^b \left( \frac{T}{\gamma} \right)^c \mathcal{H}_{c,T} \left( \alpha_{T,t} \left( l, \frac{k\gamma}{T} \right), \gamma, \frac{\rho(t\gamma h)}{T}, k; z \right). \end{aligned}$$

Now, Proposition 5.3 and Proposition 5.5 and (14) imply that

$$\begin{aligned} \mathcal{M}_T \left( u, e^{\frac{2\pi i}{k}(h+iz)} \right) &= \sum_{r=0}^{\infty} M_T^{r,\mu}(h, k; z) \frac{(2\pi i r)^r}{r!} + \sum_{r=0}^{\infty} E_T^{r,\mu}(h, k; z) \frac{(2\pi i r)^r}{r!} \\ &\quad + \sum_{r=0}^{\infty} \sum_{\substack{t=-\frac{T-1}{2} \\ t \neq 0}}^{\frac{T-1}{2}} \sum_{l=0}^{\frac{kY}{T}-1} M_T^{r,H}(t, l, h, k; z) \frac{(2\pi i r)^r}{r!} + \sum_{r=0}^{\infty} \sum_{\substack{t=-\frac{T-1}{2} \\ t \neq 0}}^{\frac{T-1}{2}} E_T^{r,H}(t, l, h, k; z) \frac{(2\pi i r)^r}{r!}, \end{aligned}$$

with certain functions  $E_T^{r,\mu}, E_T^{r,H}$ . In the variable  $z$ , the functions  $M_T^{r,\mu}, M_T^{r,H}, E_T^{r,\mu}$ , and  $E_T^{r,H}$  are functions on  $\frac{1}{i}\mathbb{H}$  with values in  $\mathbb{C}$ . The error terms  $E_T^{r,\mu}$  and  $E_T^{r,H}$  satisfy the bounds

$$\begin{aligned} (24) \quad |E_T^{2,\mu}(h, k; z)| &\ll_T |z|^{-\frac{3}{2}} e^{-\frac{\pi}{12k} \operatorname{Re}(\frac{1}{z})(\frac{1}{T}-\frac{1}{24})}, \\ |E_T^{r,\mu}(h, k; z)| &\ll_{r,T} k^{\frac{r}{2}} |z|^{-r+\frac{1}{2}} e^{-\frac{\pi}{12k} \operatorname{Re}(\frac{1}{z})(\frac{1}{T}-\frac{1}{24})}, \end{aligned}$$

and

$$\begin{aligned} (25) \quad |E_T^{2,H}(t, h, k; z)| &\ll_{r,T} k^{\frac{1}{2}} |z|^{-\frac{3}{2}} e^{-\frac{\pi}{k}(\frac{9}{4T}-\frac{1}{12}) \operatorname{Re}(\frac{1}{z})}, \\ |E_T^{r,H}(t, h, k; z)| &\ll_{r,T} k^{\frac{r}{2}} |z|^{-r+\frac{1}{2}} e^{-\frac{\pi}{k}(\frac{9}{4T}-\frac{1}{12}) \operatorname{Re}(\frac{1}{z})}. \end{aligned}$$

This implies the following result:

**Proposition 5.6.** *Let  $T > 0$  be an odd integer and suppose that  $h, k$ , and  $z$  satisfy the usual conditions. Furthermore, suppose that  $\operatorname{Re}(\frac{1}{z}) \geq \frac{k}{2}$ . Then, we have*

$$M_T^r \left( e^{\frac{2\pi i}{k}(h+iz)} \right) = M_T^{r,\mu}(h, k; z) + E_T^{r,\mu}(h, k; z) + \sum_{\substack{t=-\frac{T-1}{2} \\ t \neq 0}}^{\frac{T-1}{2}} \left( \sum_{l=0}^{\frac{kY}{T}-1} M_T^{r,H}(t, l, h, k; z) + E_T^{r,H}(t, l, h, k; z) \right),$$

where the error terms can be asymptotically bounded as in (24) and (25).

## 6. THE CIRCLE METHOD

We are now in the position to apply the Circle Method. This is a classical method and a detailed explanation can be found for example in [HW08]. For our purpose we stick closely to the general setup of [BM11] (again a more detailed exposition can be found also in the author's PhD thesis [Wal12]). We consider the  $q$ -series

$$M_T^r(q) = \sum_{n=0}^{\infty} m_T^r(n) q^n.$$

By Cauchy's Theorem, we have

$$m_T^r(n) = \frac{1}{2\pi i} \int_{\Gamma} \frac{M_T^r(q)}{q^{n+1}} dq,$$

where  $\Gamma$  is a simple counterclockwise oriented loop in the unit circle around the origin. Choosing an explicit parametrization of such a loop (depending on  $n$ ), we have

$$m_T^r(n) = \int_0^1 M_T^r \left( e^{-\frac{2\pi}{n}+2\pi i t} \right) e^{2\pi-2\pi i n t} dt.$$

Now we set  $N := \lfloor \sqrt{n} \rfloor$  and let  $\frac{h_1}{k_1} < \frac{h}{k} < \frac{h_2}{k_2}$  be adjacent Farey fractions in the Farey sequence of order  $N$ . If we define  $\Gamma_{h,k}$  to be the arc parameterized by  $t \mapsto e^{-\frac{2\pi}{n} + 2\pi it}$  for  $t$  between  $\frac{h_1+h}{k_1+k}$  and  $\frac{h+h_2}{k+k_2}$ , then the arcs  $\Gamma_{h,k}$  make up the entire circle. We find

$$(26) \quad \int_0^1 M_T^r \left( e^{-\frac{2\pi}{n} + 2\pi it} \right) e^{2\pi - 2\pi int} dt = \sum_{\substack{0 \leq h < k \leq N \\ (h,k)=1}} \int_{\Gamma_{h,k}} M_T^r \left( e^{-\frac{2\pi}{n} + 2\pi it} \right) e^{2\pi - 2\pi int} dt.$$

Substituting  $t$  by  $\phi + \frac{h}{k}$  in each integral over  $\Gamma_{h,k}$  and using the abbreviation  $z = \frac{k}{n} - ik\phi$ , we may write (26) as

$$(27) \quad m_T^r(n) = \sum_{\substack{0 \leq h < k \leq N \\ (h,k)=1}} e^{-\frac{2\pi i n h}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} M_T^r \left( e^{\frac{2\pi i}{k}(h+iz)} \right) e^{\frac{2\pi n z}{k}} d\phi,$$

where  $-\vartheta'_{h,k} := -\frac{1}{k(k_1+k)}$  and  $\vartheta''_{h,k} := \frac{1}{k(k_2+k)}$ .

Now the variable  $z$  is in the range where we can use the asymptotic formulas for  $M_T^r \left( e^{\frac{2\pi i}{k}(h+iz)} \right)$ , coming from Proposition 5.6. Hence, we can evaluate (27) by computing the integrals

$$\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} M_T^{r,\mu}(h, k; z) e^{\frac{2\pi n z}{k}} d\phi, \quad \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} M_T^{r,H}(t, l, h, k; z) e^{\frac{2\pi n z}{k}} d\phi,$$

and bounding the error integrals

$$\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} E_T^{r,\mu}(h, k; z) e^{\frac{2\pi n z}{k}} d\phi, \quad \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} E_T^{r,H}(t, h, k; z) e^{\frac{2\pi n z}{k}} d\phi.$$

**6.1. The integral over  $M_T^{r,\mu}$ .** Ignoring the factors of  $M_T^{r,\mu}(h, k; z)$  which do not depend on  $z$ , we observe that in order to compute

$$\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} M_T^{r,\mu}(h, k; z) e^{\frac{2\pi n z}{k}} d\phi$$

we are led to evaluate the integral

$$(28) \quad \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{-\frac{\pi}{12k}(z-\frac{1}{z})} z^{\frac{1}{2}-a-2c} e^{\frac{2\pi n z}{k}} d\phi.$$

In [Leh64], Lehner has shown how to evaluate this integral. We briefly outline Lehner's method, since we will proceed along similar lines when computing the integral involving  $M_T^{r,H}$ . Consider

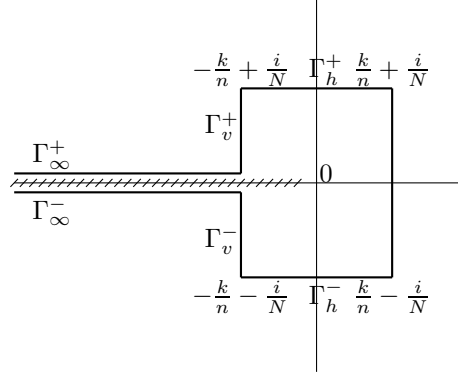
$$(29) \quad \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^d e^{\frac{\pi z}{k}(2n+\delta) + \frac{\pi}{kz}\beta} d\phi,$$

where  $\delta$  and  $\beta > 0$  are rational numbers and  $d \in \frac{1}{2} + \mathbb{Z}$ . We symmetrize the range of integration and return to the variable  $z$  through the change of variables  $\phi = \frac{iz}{k} - \frac{i}{n}$ . This shows that (29) is equal to

$$\frac{i}{k} \int_{\frac{k}{n} + \frac{i}{N}}^{\frac{k}{n} - \frac{i}{N}} z^d e^{\frac{\pi z}{k}(2n+\delta) + \frac{\pi}{kz}\beta} dz = -\frac{i}{k} \int_{\frac{k}{n} - \frac{i}{N}}^{\frac{k}{n} + \frac{i}{N}} z^d e^{\frac{\pi z}{k}(2n+\delta) + \frac{\pi}{kz}\beta} dz$$

plus an error term. After that, we again extend the range of integration to a contour  $\Gamma$  according to the following diagram in a counterclockwise manner.





This again introduces an error term. It is possible to estimate all occurring error terms and show that they are bounded by  $\frac{1}{k\sqrt{n}}(\frac{n}{k})^{|d|}$ . Now the integral over the contour in the picture above can be recognized as the Schlöfli-representation of the modified Bessel function. Indeed, according to [Wat95], page 181, we have that

$$I_\nu(x) = \frac{(\frac{x}{2})^\nu}{2\pi i} \int_\Gamma z^{-\nu-1} e^{z+\frac{x^2}{4z}} dz$$

for any  $x \in \mathbb{R}$  and  $\nu \in \mathbb{C}$ . Thus, the change of variables  $z \mapsto \frac{kz}{\pi(2n+\delta)}$  yields

**Lemma 6.1.** *Let  $n, k$  be positive integers and  $\delta$  and  $\beta > 0$  be rational numbers and  $d \in \frac{1}{2} + \mathbb{Z}$ . Then,*

$$\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^d e^{\frac{\pi z}{k}(2n+\delta) + \frac{\pi}{kz}\beta} d\phi = \frac{2\pi}{k} (2n+\delta)^{-\frac{d+1}{2}} \beta^{\frac{d+1}{2}} I_{-d-1} \left( \frac{2\pi\sqrt{\beta(2n+\delta)}}{k} \right) + \frac{1}{k\sqrt{n}} \left( \frac{n}{k} \right)^{|d|}.$$

**6.2. The integral over  $M_T^{r,H}$ .** In order to evaluate

$$\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} M_T^{r,H}(t, l, h, k; z) e^{\frac{2\pi n z}{k}} d\phi,$$

we will pursue the same approach as for  $M_T^{r,\mu}$ , i.e., we will modify the integral at the cost of introducing error terms until we can relate it to a modified Bessel function (in fact to an integral over a modified Bessel function).

If we replaced  $M_T^{r,H}(t, l, h, k; z)$  by the definition as in (23), we would have to carry along a lot of notation. Instead we first consider an integral of the following general form

$$(30) \quad \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^d e^{\frac{\pi z}{k}(2n+\delta)} e^{\frac{\beta\pi}{kz}} \mathcal{H}_{c,T}(\alpha, \gamma, \varrho, k; z) d\phi.$$

However, aiming at our later application we will put some restrictions on the parameters that will later be satisfied automatically. We will assume that  $\alpha, \beta, \gamma, \delta, \varrho$  are rational numbers, with  $|\alpha| < \frac{1}{2}$  and  $|\varrho| < \frac{1}{2}$ ,  $a$  is a positive integer and  $c$  is an integer satisfying  $c \leq r$  for some fixed integer  $r$ , and  $d \in \frac{1}{2} + \mathbb{Z}$ . We will abbreviate these properties by saying that  $\alpha, \beta, \gamma, \delta, \varrho, c$ , and  $d$  satisfy the “usual conditions”.

Furthermore, in the application later the variables  $\alpha, \beta, \gamma, \delta, \varrho$  can only attain finitely many values for fixed  $T$ . Furthermore, for fixed  $r$ , also the variables  $c$  and  $d$  can only attain finitely many values. This implies that whenever there are error terms occurring which depend on  $\alpha, \beta, \gamma, \delta, \varrho, c$ , or  $d$ , we can transform these into error terms which may be bounded solely in terms of  $T$  and  $r$ .

If one carries out our analysis in the context of [BMR12], i.e. assuming that  $T = 1$  or  $T = 3$ , then the integrals in (30) will only occur for  $\beta \leq 0$ . We first show that these integrals are small and can be

put into the error term. We give a full proof of this fact, and all will omit further proofs in the following which proceed along the same lines.

**Lemma 6.2.** *Let  $T > 0$  be an odd integer and  $r > 0$  an even integer and suppose that  $\alpha, \beta, \gamma, \delta, \varrho, c, d, h$ , and  $k$  satisfy the usual conditions. Further suppose that  $\beta \leq 0$ . Then*

$$\left| \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^d e^{\frac{\pi z}{k}(2n+\delta)} e^{\frac{\beta\pi}{kz}} \mathcal{H}_{c,T}(\alpha, \gamma, \varrho, k; z) d\phi \right| \ll_{r,T} \frac{1}{k\sqrt{n}} \left(\frac{n}{k}\right)^{|d|}.$$

*Proof.* Taking absolute values, we see that (30) is bounded by

$$\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} |z|^d e^{\frac{\pi \operatorname{Re}(z)}{k}(2n+\delta)} e^{\frac{\beta\pi}{k} \operatorname{Re}(\frac{1}{z})} |\mathcal{H}_{c,T}(\alpha, \gamma, \varrho, k; z)| d\phi.$$

We symmetrize the range of integration using that  $-\frac{1}{kN} \leq -\vartheta'_{h,k}$  and  $\vartheta''_{h,k} \leq \frac{1}{kN}$ . Then, we perform the change of variables  $\phi = \frac{iz}{k} - \frac{i}{n}$ . Thus, we obtain that (30) is bounded by

$$(31) \quad \frac{1}{k} \int_{\frac{k}{n} + \frac{i}{N}}^{\frac{k}{n} - \frac{i}{N}} |z|^d e^{\frac{\pi \operatorname{Re}(z)}{k}(2n+\delta)} e^{\frac{\beta\pi}{k} \operatorname{Re}(\frac{1}{z})} |\mathcal{H}_{c,T}(\alpha, \gamma, \varrho, k; z)| dz.$$

We now parameterize the range of integration by  $z = \frac{k}{n} - \frac{i}{N}\xi$  with  $\xi \in [-1, 1]$ . We find that  $|z| = \sqrt{\frac{k^2}{n^2} + \frac{1}{N^2}\xi^2}$ . Hence

$$\frac{k^2}{n^2} \leq |z|^2 \leq \frac{2}{n}.$$

Furthermore, we have  $\operatorname{Re}(z) = \frac{k}{n}$  and, hence,  $\operatorname{Re}(\frac{1}{z}) = \frac{\operatorname{Re}(z)}{|z|^2} \geq \frac{k}{2}$ . Using this and  $|\alpha| < \frac{1}{2}$ ,  $|\varrho| < \frac{1}{2}$  we may show that

$$|\mathcal{H}_{c,T}(\alpha, \gamma, \varrho, k; z)| \ll_{c,T} 1 \ll_{r,T} 1.$$

Plugging this into (31) and using  $\beta \leq 0$ , we see that (31) is essentially bounded (with an error term depending on  $r$  and  $T$ ) by

$$\frac{1}{k} \int_{\frac{k}{n} + \frac{i}{N}}^{\frac{k}{n} - \frac{i}{N}} \left| \frac{k}{n} \right|^d e^{\frac{\pi k}{n}(2n+\delta)} e^{\frac{\beta\pi}{k} \frac{n}{2k}} dz \ll_{r,T} \frac{1}{k\sqrt{n}} \left(\frac{n}{k}\right)^{|d|}.$$

□

Now we turn to the case  $\beta > 0$ . Our first goal is to replace the term  $e^{\frac{\beta\pi}{kz}} \mathcal{H}_{c,T}(\alpha, \gamma, \varrho, k; z)$  in (30) by an expression which can be integrated more easily. Of course this will be only possible by introducing an extra error term.

**Proposition 6.3.** *Let  $T > 0$  be an odd integer and  $r > 0$  an even integer and suppose that  $\alpha, \beta, \gamma, \delta, \varrho, c, d, h$ , and  $k$  satisfy the usual conditions. Suppose further that  $\beta > 0$ ,  $z$  satisfies the usual conditions, and, additionally, that  $\operatorname{Re}(\frac{1}{z}) \geq \frac{k}{2}$ . Then, we have*

$$e^{\frac{\beta\pi}{kz}} \mathcal{H}_{c,T}(\alpha, \gamma, \varrho, k; z) = \frac{\sqrt{\beta}\gamma}{\sqrt{T}} \int_{-1}^1 \left( \frac{\sqrt{\beta}\gamma}{\sqrt{T}} x + i\varrho \right)^c \frac{e^{-\frac{\pi\beta(x^2-1)}{kz} + 2\pi\frac{\sqrt{\beta}\gamma}{\sqrt{T}}\alpha x}}{\cosh\left(\pi\left(\frac{\sqrt{\beta}\gamma}{\sqrt{T}}x + i\varrho\right)\right)} dx + E_{\alpha,\beta,\gamma,\varrho,c,T}(z),$$

where  $E_{\alpha,\beta,\gamma,\varrho,c,T}(z)$  is a function which is bounded independently of  $z$  by a constant  $E_{r,T}$  depending only on  $r$  and  $T$ .

*Proof.* We write

$$e^{\frac{\beta\pi}{kz}} \mathcal{H}_{c,T}(\alpha, \gamma, \varrho, k; z) = e^{\frac{\beta\pi}{kz}} \int_{\mathbb{R}} (x + i\varrho)^c \frac{e^{-\frac{\pi T x^2}{\gamma^2 k z} - 2\pi x \alpha}}{\cosh(\pi(x + i\varrho))} dx.$$

We see that the qualitative behavior of the integrand (with respect to  $z$ ) changes when the sign in the exponent of  $e^{\frac{\beta\pi}{kz}} e^{-\frac{\pi T x^2}{\gamma^2 k z}}$  changes. This happens at  $\beta = -\frac{T}{\gamma^2} x^2$ . This in turn leads us to make the substitution  $x \mapsto \frac{\sqrt{\beta}\gamma}{\sqrt{T}} x$ , which yields

$$e^{\frac{\beta\pi}{kz}} \mathcal{H}_{c,T}(\alpha, \gamma, \varrho, k; z) = \frac{\sqrt{\beta}\gamma}{\sqrt{T}} \int_{\mathbb{R}} \left( \frac{\sqrt{\beta}\gamma}{\sqrt{T}} x + i\varrho \right)^c \frac{e^{-\frac{\pi\beta(x^2-1)}{kz} + 2\pi \frac{\sqrt{\beta}\gamma}{\sqrt{T}} \alpha x}}{\cosh\left(\pi \left( \frac{\sqrt{\beta}\gamma}{\sqrt{T}} x + i\varrho \right)\right)} dx.$$

We now split the integral into  $\int_{\mathbb{R}} = \int_{-\infty}^{-1} + \int_{-1}^1 + \int_1^{\infty}$ . The middle part becomes exactly the term in the statement of this proposition, while the other two integrals form an error term. Indeed, we have

$$\left| \int_1^{\infty} \left( \frac{\sqrt{\beta}\gamma}{\sqrt{T}} x + i\varrho \right)^c \frac{e^{-\frac{\pi\beta(x^2-1)}{kz} + 2\pi \frac{\sqrt{\beta}\gamma}{\sqrt{T}} \alpha x}}{\cosh\left(\pi \left( \frac{\sqrt{\beta}\gamma}{\sqrt{T}} x + i\varrho \right)\right)} dx \right| \leq \int_1^{\infty} \left| \frac{\sqrt{\beta}\gamma}{\sqrt{T}} x + i\varrho \right|^c \frac{e^{2\pi \frac{\sqrt{\beta}\gamma}{\sqrt{T}} \alpha x}}{\left| \cosh\left(\pi \left( \frac{\sqrt{\beta}\gamma}{\sqrt{T}} x + i\varrho \right)\right) \right|} dx,$$

because

$$\left| e^{-\frac{\pi\beta(x^2-1)}{kz}} \right| = e^{-\frac{\pi\beta(x^2-1)}{k} \operatorname{Re}\left(\frac{1}{z}\right)} \leq e^{-\frac{\pi\beta(x^2-1)}{2}} \leq 1.$$

The remaining integral is independent of  $z$ . Using a similar reasoning in the case  $\int_{-\infty}^{-1}$ , we finish the proof of the proposition.  $\square$

Plugging the result of Proposition 6.3 into (30), we obtain that

$$(32) \quad \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^d e^{\frac{\pi z}{k}(2n+\delta)} e^{\frac{\beta\pi}{kz}} \mathcal{H}_{c,T}(\alpha, \gamma, \varrho, k; z) d\phi \\ = \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^d e^{\frac{\pi z}{k}(2n+\delta)} \left( \frac{\sqrt{\beta}\gamma}{\sqrt{T}} \int_{-1}^1 \left( \frac{\sqrt{\beta}\gamma}{\sqrt{T}} x + i\varrho \right)^c \frac{e^{-\frac{\pi\beta(x^2-1)}{kz} + 2\pi \frac{\sqrt{\beta}\gamma}{\sqrt{T}} \alpha x}}{\cosh\left(\pi \left( \frac{\sqrt{\beta}\gamma}{\sqrt{T}} x + i\varrho \right)\right)} dx + E_{\alpha, \beta, \gamma, \varrho, c, T}(z) \right) d\phi.$$

We split this integral and evaluate the main term and the error term separately. Proceeding analogously to Lemma 6.2 one proves that the contribution from the error term is indeed small:

**Lemma 6.4.** *Let  $T > 0$  be an odd integer and  $r > 0$  an even integer and suppose that  $\alpha, \beta, \gamma, \delta, \varrho, c, d, h$ , and  $k$  satisfy the usual conditions and suppose  $\beta > 0$ . Then,*

$$\left| \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^d e^{\frac{\pi z}{k}(2n+\delta)} E_{\alpha, \beta, \gamma, \varrho, c, T}(z) d\phi \right| \ll_{r,T} \frac{1}{k\sqrt{n}} \left( \frac{n}{k} \right)^{|d|}.$$

We next consider

$$(33) \quad \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^d e^{\frac{\pi z}{k}(2n+\delta)} \left( \frac{\sqrt{\beta}\gamma}{\sqrt{T}} \int_{-1}^1 \left( \frac{\sqrt{\beta}\gamma}{\sqrt{T}} x + i\varrho \right)^c \frac{e^{-\frac{\pi\beta(x^2-1)}{kz} + 2\pi \frac{\sqrt{\beta}\gamma}{\sqrt{T}} \alpha x}}{\cosh\left(\pi \left( \frac{\sqrt{\beta}\gamma}{\sqrt{T}} x + i\varrho \right)\right)} dx \right) d\phi,$$

which is the main part of (32). We first rewrite (33) as

$$(34) \quad \lambda \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^d e^{\frac{\pi z}{k}(2n+\delta)} \int_{-1}^1 (\lambda x + i\varrho)^c \frac{e^{-\frac{\pi\beta(x^2-1)}{kz} - 2\pi\lambda\alpha x}}{\cosh(\pi(\lambda x + i\varrho))} dx d\phi,$$

where  $\lambda := \frac{\sqrt{\beta}\gamma}{\sqrt{T}}$ . In order to evaluate this integral we combine Lehner's approach [Leh64], which we sketched in the previous section, with work of Bringmann [Bri08] and Bringmann and Mahlburg [BM11] where similar integrals were considered. First, we symmetrize the range of integral in (34):

$$\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} = \int_{-\frac{1}{kN}}^{\frac{1}{kN}} - \int_{\vartheta''_{h,k}}^{\frac{1}{kN}} - \int_{-\frac{1}{kN}}^{-\vartheta'_{h,k}}.$$

Again it is rather easy to bound the second and the third integral and see that their contribution is small and will form part of the error term later.

**Lemma 6.5.** *Let  $T > 0$  be an odd integer and  $r > 0$  an even integer and suppose that  $\alpha, \beta, \gamma, \delta, \varrho, c, d, h$ , and  $k$  satisfy the usual conditions. Suppose that  $\beta > 0$ . Then, we have*

$$\lambda \int_{\vartheta''_{h,k}}^{\frac{1}{kN}} \int_{-1}^1 z^d e^{\frac{\pi z}{k}(2n+\delta)} (\lambda x + i\varrho)^c \frac{e^{-\frac{\pi\beta(x^2-1)}{kz} + 2\pi\lambda\alpha x}}{\cosh(\pi(\lambda x + i\varrho))} dx d\phi \ll_{r,T} \frac{1}{k\sqrt{n}} \left(\frac{n}{k}\right)^{|d|}$$

and

$$\lambda \int_{-\frac{1}{kN}}^{-\vartheta'_{h,k}} \int_{-1}^1 z^d e^{\frac{\pi z}{k}(2n+\delta)} (\lambda x + i\varrho)^c \frac{e^{-\frac{\pi\beta(x^2-1)}{kz} + 2\pi\lambda\alpha x}}{\cosh(\pi(\lambda x + i\varrho))} dx d\phi \ll_{r,T} \frac{1}{k\sqrt{n}} \left(\frac{n}{k}\right)^{|d|}.$$

Now we consider the main term contribution of (34) after symmetrization. We rewrite this integral using  $\phi = \frac{iz}{k} - \frac{i}{n}$ .

$$(35) \quad \begin{aligned} & \lambda \int_{-\frac{1}{kN}}^{\frac{1}{kN}} \int_{-1}^1 z^d e^{\frac{\pi z}{k}(2n+\delta)} (\lambda x + i\varrho)^c \frac{e^{-\frac{\pi\beta(x^2-1)}{kz} + 2\pi\lambda\alpha x}}{\cosh(\pi(\lambda x + i\varrho))} dx d\phi \\ &= \lambda \int_{-1}^1 \frac{(\lambda x + i\varrho)^c e^{+2\pi\lambda\alpha x}}{\cosh(\pi(\lambda x + i\varrho))} \left( -\frac{i}{k} \int_{\frac{k}{n} - \frac{i}{N}}^{\frac{k}{n} + \frac{i}{N}} z^d e^{\frac{\pi z}{k}(2n+\delta)} e^{\frac{\pi\beta(1-x^2)}{kz}} dz \right) dx. \end{aligned}$$

Our first goal is to compute the inner integral on the right hand side of (35). In order to do that we extend the range of integration as given in the diagram on page 17. Note that we have to slit the plane because  $z^d$  is a multi-valued function. On this slitted plane we use the principal branch of the logarithm. We now prove that the contribution from all the additional paths is small.

### 6.2.1. The contours $\Gamma_h^+$ and $\Gamma_h^-$ .

**Lemma 6.6.** *Uniformly for all  $x \in [-1, 1]$ , we have*

$$\left| -\frac{i}{k} \int_{\Gamma_h^+} z^d e^{\frac{\pi z}{k}(2n+\delta)} e^{\frac{\pi\beta(1-x^2)}{kz}} \right| \ll_T 1.$$

*The same is true for the integral over  $\Gamma_h^-$ .*

*Proof.* In the range of integration one has  $|z|^d \leq n^{-\frac{|d|}{2}}$ ,  $\operatorname{Re}(z) \leq \frac{k}{n}$  and  $\operatorname{Re}\left(\frac{1}{z}\right) = \frac{\operatorname{Re}(z)}{|z|^2} \leq k$ . We conclude that

$$\left| \int_{\Gamma_h^+} z^d e^{\frac{\pi z}{k}(2n+\delta)} e^{\frac{\pi\beta(1-x^2)}{kz}} dz \right| \leq \int_{\Gamma_h^+} |z|^d e^{\frac{\pi}{k}(2n+\delta)\operatorname{Re}(z)} e^{\frac{\pi\beta(1-x^2)}{k}\operatorname{Re}\left(\frac{1}{z}\right)} dz \leq \int_{\Gamma_h^+} n^{-\frac{|d|}{2}} e^{\pi(2+\frac{\delta}{n})} e^{\pi\beta(1-x^2)} dz.$$

This integral can be bounded by a constant only depending on  $T$  uniformly in  $x$ .  $\square$

### 6.2.2. The contours $\Gamma_v^+$ and $\Gamma_v^-$ .

**Lemma 6.7.** *Uniformly for all  $x \in [-1, 1]$ , we have*

$$\left| -\frac{i}{k} \int_{\Gamma_v^+} z^d e^{\frac{\pi z}{k}(2n+\delta)} e^{\frac{\pi\beta(1-x^2)}{kz}} dz \right| \ll_{d,T} \frac{1}{k\sqrt{n}} \left(\frac{n}{k}\right)^{|d|}.$$

The same estimate holds for the integral over  $\Gamma_v^-$ .

*Proof.* Here, in the range of integration, we have  $|z|^d \leq \left(\frac{n}{k}\right)^{|d|}$ . Of course, the real parts of both  $z$  and  $\frac{1}{z}$  are negative on  $\Gamma_v^+$ . Hence, we obtain

$$\left| -\frac{i}{k} \int_{\Gamma_v^+} z^d e^{\frac{\pi z}{k}(2n+\delta)} e^{\frac{\pi\beta(1-x^2)}{kz}} dz \right| \leq \frac{1}{k} \int_{\Gamma_v^+} |z|^d \underbrace{e^{\frac{\pi}{k}(2n+\delta)\operatorname{Re}(z)}}_{\leq 1} \underbrace{e^{\frac{\pi\beta(1-x^2)}{k}\operatorname{Re}\left(\frac{1}{z}\right)}}_{\leq 1} dz \ll_{d,T} \frac{1}{k\sqrt{n}} \left(\frac{n}{k}\right)^{|d|}.$$

$\square$

### 6.2.3. The contours $\Gamma_\infty^+$ and $\Gamma_\infty^-$ .

**Lemma 6.8.** *Uniformly for all  $x \in [-1, 1]$ , we have*

$$\left| -\frac{i}{k} \int_{\Gamma_\infty^+} z^d e^{\frac{\pi z}{k}(2n+\delta)} e^{\frac{\pi\beta(1-x^2)}{kz}} dz \right| \ll_{d,T} \frac{1}{k\sqrt{n}} \left(\frac{n}{k}\right)^{|d|}.$$

The same estimate holds for the integral over  $\Gamma_\infty^-$ .

*Proof.* We find

$$\int_{\Gamma_\infty^+} z^d e^{\frac{\pi z}{k}(2n+\delta)} e^{\frac{\pi\beta(1-x^2)}{kz}} dz = \int_{-\frac{k}{N}}^{-\infty} e^{d \log z} e^{\frac{\pi z}{k}(2n+\delta)} e^{\frac{\pi\beta(1-x^2)}{kz}} dz,$$

and, similarly,

$$\int_{\Gamma_\infty^-} z^d e^{\frac{\pi z}{k}(2n+\delta)} e^{\frac{\pi\beta(1-x^2)}{kz}} dz = e^{-2\pi i d} \int_{-\infty}^{-\frac{k}{N}} e^{d(\log z - 2\pi i)} e^{\frac{\pi z}{k}(2n+\delta)} e^{\frac{\pi\beta(1-x^2)}{kz}} dz.$$

In either case it remains to bound

$$(36) \quad \int_{-\infty}^{-\frac{k}{N}} e^{d \log z} e^{\frac{\pi z}{k}(2n+\delta)} e^{\frac{\pi\beta(1-x^2)}{kz}} dz.$$

Substituting  $z \mapsto -\frac{1}{z}$ , we find that (36) is bounded by

$$\int_0^{\frac{N}{k}} |z|^{-d-2} e^{-\frac{\pi}{k}(2n+\delta)\operatorname{Re}\left(\frac{1}{z}\right)} e^{\frac{\pi\beta(x^2-1)}{k}\operatorname{Re}(z)} dz = \int_0^{\frac{N}{k}} |z|^{-d-2} e^{-\frac{\pi}{kz}(2n+\delta)} e^{\frac{\pi\beta(x^2-1)z}{k}} dz.$$

Using that  $x \in [-1, 1]$  and that  $z \in \mathbb{R}^+$  in the range of integration, we see that  $e^{\frac{\pi\beta(x^2-1)z}{k}} \leq 1$ . Thus, (36) is bounded by

$$\int_0^{\frac{N}{k}} |z|^{-d-2} e^{-\frac{\pi}{kz}(2n+\delta)} dz.$$

Now we distinguish two cases. If  $-d-2 < 0$ , then we see that the function

$$z \mapsto |z|^{-d-2} e^{-\frac{\pi}{kz}(1+\delta)}$$

is bounded for  $z \in \mathbb{R}^+$  by a constant depending on  $d$ . Furthermore, the function  $z \mapsto e^{-\frac{\pi}{zk}(2n-1)}$  is monotonically increasing in  $z$ . Thus, it attains its maximal value at the boundary at  $\frac{N}{k}$ , namely  $e^{-\frac{\pi}{N}(2n-1)} \leq 1$ . We may now assume that  $-d-2 \geq 0$ . In this case the function  $z \mapsto |z|^{-d-2} e^{-\frac{\pi}{kz}(2n+\delta)}$  is monotonically increasing and we may again evaluate at  $\frac{N}{k}$ , in which case the integral is bounded by

$$\left(\frac{N}{k}\right)^{-d-2} = \frac{1}{\sqrt{n}} \left(\frac{n}{k}\right)^{|d|} n^{-|d|/2 - \frac{1}{2}} k^2.$$

Using the fact that  $|d| \geq 2$ , completes the proof of the lemma.  $\square$

6.2.4. *Schl\"afli's integral and the evaluation.* Combining Lemmas 6.7, 6.6, and 6.8, we see that

$$-\frac{i}{k} \int_{\frac{k}{n} - \frac{i}{N}}^{\frac{k}{n} + \frac{i}{N}} z^d e^{\frac{\pi z}{k}(2n+\delta)} e^{\frac{\pi\beta(1-x^2)}{kz}} dz = -\frac{i}{k} \int_{\Gamma} z^d e^{\frac{\pi z}{k}(2n+\delta)} e^{\frac{\pi\beta(1-x^2)}{kz}} dz + E_{T,\delta,\beta}(d, k, n; x),$$

where the error term satisfies

$$|E_{T,\delta,\beta}(d, k, n; x)| \ll_{d,T} \frac{1}{k\sqrt{n}} \left(\frac{n}{k}\right)^{|d|}$$

uniformly for all  $x \in [-1, 1]$ . As in Lemma 6.1, we use the Schl\"afli integral representation given on page 181 of [Wat95] to see that

$$-\frac{i}{k} \int_{\Gamma} z^d e^{\frac{\pi z}{k}(2n+\delta)} e^{\frac{\pi\beta(1-x^2)}{kz}} dz = \frac{2\pi}{k} (2n+\delta)^{-\frac{d+1}{2}} \beta^{\frac{d+1}{2}} (1-x^2)^{\frac{d+1}{2}} I_{-d-1} \left( \frac{2\pi\sqrt{\beta(1-x^2)(2n+\delta)}}{k} \right).$$

Inserting this into (35), we obtain

$$\begin{aligned} & \lambda \int_{-\frac{1}{kN}}^{\frac{1}{kN}} \int_{-1}^1 z^d e^{\frac{\pi z}{k}(2n+\delta)} (\lambda x + i\rho)^c \frac{e^{-\frac{\pi\beta(x^2-1)}{kz} - 2\pi\lambda\alpha x}}{\cosh(\pi(\lambda x + i\rho))} dx d\phi \\ &= \lambda \int_{-1}^1 \frac{(\lambda x + i\rho)^c e^{2\pi\lambda\alpha x}}{\cosh(\pi(\lambda x + i\rho))} \frac{2\pi}{k} (2n+\delta)^{-\frac{d+1}{2}} \beta^{\frac{d+1}{2}} (1-x^2)^{\frac{d+1}{2}} I_{-d-1} \left( \frac{2\pi\sqrt{\beta(1-x^2)(2n+\delta)}}{k} \right) dx \\ & \quad + \lambda \int_{-1}^1 \frac{(\lambda x + i\rho)^c e^{2\pi\lambda\alpha x}}{\cosh(\pi(\lambda x + i\rho))} E_{T,\delta,\beta}(d, k, n; x) dx. \end{aligned}$$

Finally, one shows that the last summand is again of the size of the error terms, which we obtained so far. This allows us now to give the final formula for (30). In order to state our result more succinctly, we introduce the abbreviation

$$(37) \quad \mathcal{I}_{T;\alpha,\beta,\delta,\rho}(c, d, k; n) := \int_{-1}^1 \frac{\left(\frac{\sqrt{\beta}\gamma}{\sqrt{T}}x + i\rho\right)^c e^{2\pi\frac{\sqrt{\beta}\gamma}{\sqrt{T}}\alpha x}}{\cosh\left(\pi\left(\frac{\sqrt{\beta}\gamma}{\sqrt{T}}x + i\rho\right)\right)} (1-x^2)^{\frac{d+1}{2}} I_{-d-1} \left( \frac{2\pi\sqrt{\beta(1-x^2)(2n+\delta)}}{k} \right) dx.$$

Then, combining Lemma 6.4, 6.5, 6.6, 6.7 and 6.8 we obtain the following expression for (30):

**Proposition 6.9.** *Let  $T > 0$  be an odd integer,  $r > 0$  an even integer and suppose that  $\alpha, \beta, \gamma, \delta, \varrho, c, d, h$ , and  $k$  satisfy the usual conditions. Further suppose that  $\beta > 0$ . Then*

$$\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^d e^{\frac{\pi z}{k}(2n+\delta)} e^{\frac{\beta \pi}{kz}} \mathcal{H}_{c,T}(\alpha, \gamma, \varrho, k; z) d\phi = \frac{2\pi\gamma}{k\sqrt{T}} (2n+\delta)^{-\frac{d+1}{2}} \beta^{\frac{d}{2}+1} \mathcal{I}_{T;\alpha,\beta,\delta,\varrho}(c, d, k; n) + O_{r,T} \left( \frac{1}{k\sqrt{n}} \left( \frac{n}{k} \right)^{|d|} \right).$$

**6.3. Integrating the errors.** Our last task in this section will be to bound the integrals

$$\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} E_T^{r,H}(t, h, k; z) e^{\frac{2\pi n z}{k}} d\phi \quad \text{and} \quad \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} E_T^{r,\mu}(h, k; z) e^{\frac{2\pi n z}{k}} d\phi.$$

We see that at this point we have to put a restriction on  $T$ .

**Lemma 6.10.** *Let  $h, k$  satisfy the usual conditions. Suppose that  $0 < T < 27$  is an odd integer. Then,*

$$\left| \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} E_T^{2,H}(t, l, h, k; z) e^{\frac{2\pi n z}{k}} d\phi \right| \ll_T n k^{-2} \quad \text{and} \quad \left| \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} E_T^{r,H}(t, l, h, k; z) e^{\frac{2\pi n z}{k}} d\phi \right| \ll_{r,T} n^{r-1} k^{-\frac{r}{2}-\frac{1}{2}}.$$

*Proof.* We only treat the case  $r > 2$ , and  $r = 2$  is proven similarly. Using Proposition 5.6, taking absolute values and extending the range of integration, we see that

$$\left| \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} E_T^{r,H}(t, l, h, k; z) e^{\frac{2\pi n z}{k}} d\phi \right| \ll_{r,T} \frac{1}{k} \int_{\frac{k}{n} + \frac{i}{N}}^{\frac{k}{n} - \frac{i}{N}} k^{\frac{r}{2}} |z|^{-r+\frac{1}{2}} e^{-\frac{\pi}{k}(\frac{9}{4T}-\frac{1}{12})\text{Re}(\frac{1}{z})} dz.$$

As in Lemma 6.2, we observe that in the given range of integration we have  $\text{Re}(z) = \frac{k}{n}$  and  $\text{Re}(\frac{1}{z}) = \frac{\text{Re}(z)}{|z|^2} \geq \frac{k}{2}$ . If  $0 < T < 27$ , the sign in the exponent is negative. Hence, we obtain

$$\left| \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} E_T^{r,H}(t, l, h, k; z) e^{\frac{2\pi n z}{k}} d\phi \right| \ll_{r,T} \frac{1}{k} \int_{\frac{k}{n} + \frac{i}{N}}^{\frac{k}{n} - \frac{i}{N}} k^{\frac{r}{2}} \left| \frac{k}{n} \right|^{-r+\frac{1}{2}} e^{-\frac{\pi}{k}(\frac{9}{4T}-\frac{1}{12})\frac{k}{2}} dz \ll_{r,T} n^{r-1} k^{-\frac{r}{2}-\frac{1}{2}}.$$

□

**Lemma 6.11.** *Let  $h, k$  satisfy the usual conditions. Suppose that  $0 < T < 24$  is an odd integer. Then,*

$$\left| \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} E_T^{2,\mu}(h, k; z) e^{\frac{2\pi n z}{k}} d\phi \right| \ll_T n k^{-\frac{5}{2}} \quad \text{and} \quad \left| \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} E_T^{r,\mu}(h, k; z) e^{\frac{2\pi n z}{k}} d\phi \right| \ll_{r,T} k^{-\frac{r}{2}-\frac{1}{2}}.$$

## 7. PROOF OF THE MAIN THEOREMS

**7.1. Proof of Theorem A.** Recall that (27) states that

$$m_T^r(n) = \sum_{\substack{0 \leq h < k \leq N \\ (h,k)=1}} e^{-\frac{2\pi i n h}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} M_T^r \left( e^{\frac{2\pi i}{k}(h+iz)} \right) e^{\frac{2\pi n z}{k}} d\phi.$$

Furthermore, recall the representation

$$M_T^r \left( e^{\frac{2\pi i}{k}(h+iz)} \right) = M_T^{r,\mu}(h, k; z) + E_T^{r,\mu}(h, k; z) + \sum_{\substack{t=-\frac{T-1}{2} \\ t \neq 0}}^{\frac{T-1}{2}} \left( \sum_{l=0}^{\frac{k\gamma}{T}-1} M_T^{r,H}(t, l, h, k; z) + E_T^{r,H}(t, l, h, k; z) \right)$$

given in Proposition 5.6. Combining these equations with the integral evaluations and estimates from the previous section, we now derive the asymptotic formula for  $m_T^r(n)$  as given in Theorem A. We treat the contributions coming from  $M_T^{r,\mu}$ ,  $M_T^{r,H}$ ,  $E_T^{r,\mu}$ , and  $E_T^{r,H}$  separately.

To describe the contribution coming from  $M_T^{r,\mu}$ , we require the Kloosterman sum

$$(38) \quad K_k(n) := -i^{\frac{3}{2}} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} e^{-\frac{2\pi i n h}{k}} e^{\frac{\pi i}{12k}(h-[h]_k)} \chi^{-1}(h, [-h]_k, k).$$

**Proposition 7.1.** *We have*

$$\begin{aligned} & \sum_{\substack{0 \leq h < k \leq N \\ (h,k)=1}} e^{-\frac{2\pi i n h}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} M_T^{r,\mu}(h, k; z) e^{\frac{2\pi n z}{k}} d\phi \\ &= 2\pi \sum_{k \leq \sqrt{n}} \frac{K_k(n)}{k} \sum_{2a+2b+2c=r} \kappa(a, b, c) (kT)^a (24n-1)^{-\frac{3}{4}+\frac{a}{2}+c} I_{-\frac{3}{2}+a+2c} \left( \frac{\pi \sqrt{24n-1}}{6k} \right) + O_{r,T}(n^{r-1}). \end{aligned}$$

*Proof.* Using (22), we see that the contribution coming from  $M_T^{r,\mu}$  is given by

$$\begin{aligned} & \sum_{\substack{0 \leq h < k \leq N \\ (h,k)=1}} e^{-\frac{2\pi i n h}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} M_T^{r,\mu}(h, k; z) e^{\frac{2\pi n z}{k}} d\phi = -i^{\frac{3}{2}} \sum_{\substack{0 \leq h < k \leq N \\ (h,k)=1}} e^{-\frac{2\pi i n h}{k}} e^{\frac{\pi i}{12k}(h-[h]_k)} \chi^{-1}(h, [-h]_k, k) \\ & \times \sum_{2a+2b+2c=r} \kappa(a, b, c) (kT)^a \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{-\frac{\pi}{12k}(z-\frac{1}{z})} z^{\frac{1}{2}-a-2c} e^{\frac{2\pi n z}{k}} d\phi. \end{aligned}$$

Using Lemma 6.1 and the definition of the Kloosterman sum, we immediately obtain the main term. Hence, we are left with estimating the error term. After taking absolute values and using Lemma 6.1, we see that the error term is essentially bounded (in terms of  $r$  and  $T$ ) by

$$\sum_{\substack{0 \leq h < k \leq N \\ (h,k)=1}} \sum_{2a+2b+2c=r} \kappa(a, b, c) (kT)^a \frac{1}{k\sqrt{n}} \left( \frac{n}{k} \right)^{|\frac{1}{2}-a-2c|} \ll_{r,T} \sum_{k \leq N} \sum_{2a+2b+2c=r} n^{|\frac{1}{2}-a-2c|-\frac{1}{2}} k^{a-|\frac{1}{2}-a-2c|}.$$

We distinguish the contributions to the error term coming from two subcases. Either  $a = b = 0$  and  $2c = r$ , and, hence,

$$n^{|\frac{1}{2}-a-2c|-\frac{1}{2}} k^{a-|\frac{1}{2}-a-2c|} = n^{r-1} k^{-|\frac{1}{2}-r|}.$$

In this case, the contribution to the error term is  $O_{r,T}(n^{r-1})$ . In the other case we find

$$n^{|\frac{1}{2}-a-2c|-\frac{1}{2}} k^{a-|\frac{1}{2}-a-2c|} \leq n^{r-2} k^{\frac{1}{2}}.$$

Hence, the contribution to the error term is of size

$$\sum_{k \leq N} n^{|\frac{1}{2}-a-2c|-\frac{1}{2}} k^{a-|\frac{1}{2}-a-2c|} \ll_{r,T} \sum_{k \leq N} n^{r-2} k^{\frac{1}{2}} \ll n^{r-2} N^{\frac{3}{2}} \ll n^{r-1}.$$

□

We next consider

$$\sum_{\substack{0 \leq h < k \leq N \\ (h,k)=1}} e^{-\frac{2\pi i n h}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} \sum_{t=-\frac{T-1}{2}}^{\frac{T-1}{2}} \sum_{\substack{l=0 \\ l \neq 0}}^{\frac{k\gamma}{T}-1} M_T^{r,H}(t, l, h, k; z) e^{\frac{2\pi n z}{k}} d\phi,$$



which is the contribution for  $m_T^r(n)$  coming from  $M_T^{r,H}$ . We define the partial Kloosterman sum

$$(39) \quad K_{\sigma,\varrho,l;k}(n) := \sum_{\substack{0 \leq h < k \\ (h,k)=1 \\ \rho\left(\frac{Th}{\sigma}\right)=\varrho}} e^{-\frac{2\pi i n h}{k}} U_H^*(T, t, l, h, k).$$

Now we proceed as before in the case of  $M_T^{r,\mu}$  and use the integral evaluations from the previous section and bound the occurring error terms. Then, our result can be stated as follows:

**Proposition 7.2.** *We find that*

$$\begin{aligned} & \sum_{\substack{0 \leq h < k \leq N \\ (h,k)=1}} e^{-\frac{2\pi i n h}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} \sum_{t=-\frac{T-1}{2}, t \neq 0}^{\frac{T-1}{2}} \sum_{l=0}^{\frac{k\gamma}{T}-1} M_T^{r,H}(t, l, h, k; z) e^{\frac{2\pi n z}{k}} d\phi \\ &= 2\pi \sum_{\substack{\sigma|T \\ t \neq 0}} \sum_{t=-\frac{T-1}{2}}^{\frac{T-1}{2}} \sum_{\varrho=-\frac{T-1}{2}}^{\frac{T-1}{2}} \sum_{\substack{0 < k \leq N \\ (k,T)=\sigma}} \sum_{l=0}^{\frac{k}{\sigma}-1} \frac{K_{\sigma,\varrho,l;k}(n)}{k} \sum_{2a+(2b+1)+c=r} \kappa^*(a, b, c) k^{b-\frac{1}{2}} T^{b-\frac{1}{2}} \sigma^{c+\frac{1}{2}} \left(2n - \frac{1}{12}\right)^{\frac{a+c}{2}-\frac{1}{4}} \\ & \times \left(\frac{1}{12} - \frac{\sigma^2}{T^3} \left(\varrho^2 + \frac{T^2}{4} - |\varrho|T\right)\right)_+^{\frac{3}{4}-\frac{a+c}{2}} \mathcal{I}_{T;\alpha_T,t(l,\frac{k}{\sigma}),\frac{1}{12}-\frac{\sigma^2}{T^3}\left(\varrho^2+\frac{T^2}{4}-|\varrho|T\right),-\frac{1}{12},\frac{\varrho}{T}} \left(c, -\frac{1}{2} - a - c, k; n\right) + E_{r,T}(n). \end{aligned}$$

Here,  $E_{r,T}(n)$  is an error term. If  $r = 2$ , then the error term has the magnitude  $O_T(n \log n)$ , whereas it is of order  $O_{r,T}(n^{r-1})$  if  $r > 2$ .

Finally, we use Lemmas 6.10 and 6.11 to prove the following bounds on the error terms.

**Lemma 7.3.** *Let  $0 < T < 24$  be an odd integer. Then*

$$\left| \sum_{\substack{0 \leq h < k \leq N \\ (h,k)=1}} e^{-\frac{2\pi i n h}{k}} \sum_{\substack{t=-\frac{T-1}{2} \\ t \neq 0}}^{\frac{T-1}{2}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} E_T^{2,H}(t, l, h, k; z) d\phi \right| \ll_{r,T} n \log n,$$

and, for  $r > 2$ , we have

$$\sum_{\substack{0 \leq h < k \leq N \\ (h,k)=1}} e^{-\frac{2\pi i n h}{k}} \sum_{\substack{t=-\frac{T-1}{2} \\ t \neq 0}}^{\frac{T-1}{2}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} E_T^{r,H}(t, l, h, k; z) d\phi \ll_{r,T} n^{r-1}.$$

Furthermore, for all  $r \geq 2$  even, we have

$$\sum_{\substack{0 \leq h < k \leq N \\ (h,k)=1}} e^{-\frac{2\pi i n h}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} E_T^{r,\mu}(h, k; z) d\phi \ll_{r,T} n^{r-1}.$$

*Proof of Theorem A.* Proposition 7.1, Proposition 7.2, and Lemma 7.3 prove Theorem A.  $\square$

**7.2. Proof of Theorem B.** We now want to identify the leading term in Theorem A. For this purpose we have to find asymptotic expressions for both the modified Bessel functions and the integrals over modified Bessel functions appearing in Theorem A. In fact, it will turn out that the contributions coming from the integrals over the modified Bessel functions are smaller than those of the modified Bessel functions themselves. Thus, we only have to find a precise description of the former and bound the latter. The key ingredient for both parts is the following well-known approximation of the Bessel function which follows from [OLBC10] 10.40(i). As  $y \rightarrow \infty$ , we have

$$(40) \quad I_\nu(y) = \frac{e^y}{\sqrt{2\pi y}} + O\left(y^{-\frac{3}{2}} e^y\right).$$

**Proposition 7.4.** *Let  $T < 24$  be an odd integer and  $r$  an even integer. Then, we have*

$$(41) \quad \begin{aligned} & 2\pi \sum_{k \leq \sqrt{n}} \frac{K_k(n)}{k} \sum_{2a+2b+2c=r} \kappa(a, b, c) (kT)^a (24n-1)^{-\frac{3}{4}+\frac{a}{2}+c} I_{-\frac{3}{2}+a+2c} \left( \frac{\pi\sqrt{24n-1}}{6k} \right) \\ &= 2\sqrt{3}(-1)^{\frac{r}{2}} B_r\left(\frac{1}{2}\right) (24n)^{\frac{r}{2}-1} e^{\pi\sqrt{\frac{2n}{3}}} + O_{r,T} \left( n^{\frac{r}{2}-\frac{3}{2}} e^{\pi\sqrt{\frac{2n}{3}}} \right). \end{aligned}$$

*Proof.* Equation (40) easily implies that the arguments of the modified Bessel functions play the decisive role with regards to the asymptotic behavior of (41). We see that the leading contribution must come from  $k = 1$ . All terms coming from  $k > 1$  are even exponentially smaller than the error term in the  $k = 1$  approximation for the modified Bessel function. As a result we can omit all these terms and place them in the error term. By (40) we obtain

$$I_{-\frac{3}{2}+a+2c} \left( \frac{\pi\sqrt{24n-1}}{6} \right) = \frac{\sqrt{3}}{\pi(24n)^{\frac{1}{4}}} e^{\pi\sqrt{\frac{2n}{3}}} + O\left( \frac{1}{n^{\frac{3}{4}}} e^{\pi\sqrt{\frac{2n}{3}}} \right).$$

Furthermore, observe that  $(24n-1)^{-\frac{3}{4}+\frac{a}{2}+c}$  is maximized if  $2c = r$ . In this case we obtain the expression  $(24n-1)^{-\frac{3}{4}+\frac{r}{2}}$ . The next smaller term comes from  $2a = 2$  and  $2c = r-2$ , in this case we obtain  $(24n-1)^{-\frac{3}{4}+\frac{r}{2}-\frac{1}{2}}$ . This summand and all others corresponding to the possible choices of  $a, b, c$  will also contribute to the error term with a size smaller than the error term coming from the approximation of the modified Bessel function itself. Bounding  $\kappa(a, b, c)$  by a constant depending on  $r$  and  $T$ , we find that

$$\begin{aligned} & \sum_{2a+2b+2c=r} \kappa(a, b, c) T^a (24n-1)^{-\frac{3}{4}+\frac{a}{2}+c} I_{-\frac{3}{2}+a+2c} \left( \frac{\pi\sqrt{24n-1}}{6} \right) \\ &= \frac{\sqrt{3}}{\pi} \kappa\left(0, 0, \frac{r}{2}\right) (24n)^{\frac{r}{2}-1} e^{\pi\sqrt{\frac{2n}{3}}} + O_{r,T} \left( n^{\frac{r}{2}-\frac{3}{2}} e^{\pi\sqrt{\frac{2n}{3}}} \right). \end{aligned}$$

Observing that  $K_1(n) = 1$  and  $\kappa\left(0, 0, \frac{r}{2}\right) = (-1)^{\frac{r}{2}} B_r\left(\frac{1}{2}\right)$ , we conclude the proof of this proposition.  $\square$

As a next step we need to find an asymptotic bound for the function

$$\mathcal{I}_{T; \alpha T, t(l, \frac{k}{\sigma}), \frac{1}{12} - \frac{\sigma^2}{T^3} (\varrho^2 + \frac{T^2}{4} - |\varrho|T), -\frac{1}{12}, \frac{\varrho}{T}} \left( c, -\frac{1}{2} - a - c, k; n \right),$$

which appears in Theorem A. For simplicity we treat the general case. Recall the definition (37).

**Proposition 7.5.** *Let  $T > 0$  be an odd integer and  $r > 0$  an even integer and suppose that  $\alpha, \beta, \delta, \varrho, c, d, h$ , and  $k$  satisfy the usual conditions. Further suppose that  $\beta > 0$  and  $d + \frac{1}{2} \leq 0$ . Then, we have*

$$\mathcal{I}_{T; \alpha, \beta, \delta, \varrho}(c, d, k; n) = O_{r,T} \left( n^{\frac{d}{2} + \frac{1}{4}} e^{\frac{2\pi\sqrt{\beta 2n}}{k}} \right).$$

*Proof.* In order to find the bounds for  $\mathcal{I}$ , we cannot simply plug in the approximation for the modified Bessel function (as in (40)) into equation (37) and then estimate the resulting integral. This is due to the fact that the approximation in (40) only gives reasonable bounds for sufficiently large  $y$ . However, as  $y \rightarrow 0$ , the function  $I_\nu(y)$  decays if  $\nu > 0$ . This is not accounted for in (40).

For that reason we split the integral for  $\mathcal{I}_{T;\alpha,\beta,\delta,\varrho}(c, d, k; n)$  according to whether the argument of the modified Bessel function in (37) is smaller or bigger than 1. We abbreviate the corresponding integrals as  $\int_{>1}$  and  $\int_{<1}$ . Using the approximation for the modified Bessel function (40) we find

$$\left| \int_{>1} \right| \ll_{r,T} n^{\frac{d}{2} + \frac{1}{4}} e^{\frac{2\pi\sqrt{\beta 2n}}{k}}.$$

For the other case  $\int_{<1}$  we use [OLBC10] 10.30(i), to find that  $I_\nu(y) \sim \frac{1}{\Gamma(\nu+1)} \left(\frac{y}{2}\right)^\nu$ , as  $y \rightarrow 0$ . Using this approximation of the modified Bessel function, one can easily show that

$$\left| \int_{<1} \right| \ll_{r,T} n^{-\frac{d+1}{2}}.$$

□

*Proof of Theorem B.* We only show the first assertion. The second one can be dealt with similarly.

First, we observe that term appearing in the statement of Theorem B is exactly the leading term in Proposition 7.4. Hence, it suffices to show that all contributions in Theorem A involving the functions

$$(42) \quad \mathcal{I}_{T;\alpha_{T,t}(l, \frac{k}{\sigma}), \frac{1}{12} - \frac{\sigma^2}{T^3} \left( \varrho^2 + \frac{T^2}{4} - |\varrho|T \right), -\frac{1}{12}, \frac{\varrho}{T}} \left( c, -\frac{1}{2} - a - c, k; n \right)$$

are small. Indeed, looking at the result of Proposition 7.5, we see that (42) is bounded by  $n^{-\frac{a+c}{2}} e^{\frac{2\pi\sqrt{\beta 2n}}{k}}$ , where

$$\beta := \frac{1}{12} - \frac{\sigma^2}{T^3} \left( \varrho^2 + \frac{T^2}{4} - |\varrho|T \right).$$

Hence, by Proposition 7.4, we see that (42) is smaller than the error term appearing in Proposition 7.4 if we can show that  $\beta < \frac{1}{12}$ . This is equivalent to showing that  $\frac{\sigma^2}{T^3} \left( \varrho^2 + \frac{T^2}{4} - |\varrho|T \right) > 0$ . To see that this is indeed true, first observe that  $|\varrho| \leq \frac{T-1}{2}$ . In this range the function  $\varrho \mapsto \varrho^2 - |\varrho|T + \frac{T^2}{4}$  attains its minimum at the boundary  $\varrho = \pm \frac{T-1}{2}$  with value  $\left(\frac{T-1}{2}\right)^2 - \frac{T-1}{2}T + \frac{T^2}{4} = \frac{1}{4}$ . □

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